Advanced Algebra II

Homework 10

due on May. 21, 2004

k is an algebraically closed field unless otherwise stated.

(1) Consider the ring homomorphism $\varphi : k[x, y, z, w] \to k[s, t]$ by $x \mapsto s^3, y \mapsto s^2t, z \mapsto st^2, w \mapsto t^3$. Determine Ker$\varphi$. Is Ker$\varphi$ a prime ideal?

(2) Determine all prime ideal of $k[x, y]$.

(3) Let R be a ring. And let Spec(R) be the set of all prime ideals of R. For an ideal $I \lhd R$, we define

$$\mathcal{V}(I) := \{\mathfrak{p} \in \mathrm{Spec(R)} | \mathrm{I} \subset \mathfrak{p}\}.$$

Show that we can define the "Zariski topology" on Spec(R) by considering $\mathcal{V}(I)$ as closed sets.

(4) Consider $R := k[x, y]/(x^n - y^m)$, where $(n, m) = 1$. Show that $(x^n - y^m)$ is prime. Find an algebraically independent element $t \in R$ such that R is integral over $k[t]$.

Moreover, let's define the degree of the extension $R/k[t]$ to be $[K : k(t)]$, where K denote the quotient field of R. What's the degree of your extension $R/k[t]$? What's the minimum possible degree?