Jan. 12, 2007

4.4. **injective.** In this section, we are going to define injective objects. Then one has injective resolution if the category has enough injectives. Moreover, we will see that injective resolution are convenient for handling left exact but not exact functors.

Definition 4.4.1. *Let $\mathcal{A}$ be an abelian category. An object $I \in \mathcal{A}$ is injective if for all $0 \to A \to B$ and $A \to I$, there exists $B \to I$ makes the diagram commute.*

Proposition 4.4.2. *I is injective if and only if the functor $M \mapsto Hom_{\mathcal{A}}(M, I)$ is exact.*

Proof. For every exact sequence $0 \to A \to B \to C \to 0$, we have exact sequence

$$\mathrm{Hom}(A, I) \leftarrow \mathrm{Hom}(B, I) \leftarrow \mathrm{Hom}(C, I) \leftarrow 0.$$

The definition of injective says nothing more than that $\mathrm{Hom}(B, I) \to \mathrm{Hom}(A, I)$ is surjective. □

Exercise 4.4.3. *If I is injective, then every sequence $0 \to I \to B \to C \to 0$ splits.*

An abelian category $\mathcal{A}$ is said to have **enough injectives** if for every $A \in \mathcal{A}$, there exist an injective object $I \in \mathcal{A}$ and an injection $0 \to A \to I$.

Suppose now that $\mathcal{A}$ has enough injectives. Then for every $A \in A$, one has $0 \to A \xrightarrow{\imath} I^0$ for some injective I^0. Next look at $\mathrm{coker}(\imath)$, one has $0 \to \mathrm{coker}(\imath) \to I^1$ for some injective I^1 and let $d^0 : I^0 \to I^1$ be the composition map. Inductively, we obtained a sequence

$$0 \to A \to I^0 \to I^1 ...$$

It's easy to see that it's an exact sequence because it patches short exact sequences $0 \to \mathrm{coker}(\imath_{j-1}) \xrightarrow{\imath_j} I^j \to \mathrm{coker}(\imath_j) \to 0$.

Before we move on, it worthwhile to think what indeed injective object is and why we expect an abelian category has enough injectives.

Let *Ab* be the abelian category of abelian groups. A group G is said to be **divisible** if $m : G \to G$ by $m : x \mapsto mx$ is surjective for all $m \neq 0 \in \mathbb{Z}$. In other words, for $x \in G$, and for all $m \neq 0 \in \mathbb{Z}$, there is $y \in G$ such that $ny = x$. We will show that in *Ab*:

Lemma 4.4.4. *G is divisible, then G is injective.*

Lemma 4.4.5. *Every abelian group can be embedded into a divisible group.*

Thus the abelian category *Ab* has enough injective. It also follows that those natural abelian categories, such as category of R-modules, category of sheaves of abelian groups, has enough injective.

In order to prove the Lemmata, we observe that:

(1) if G is divisible, so if G/N for any normal subgroup N.
(2) if G_i are divisible for all i, then $\sum_{i\in I} G_i$ is divisible.

proof of 4.4.4. Suppose that G is divisible and $0 \to A' \to A$ is exact with a map $f' : A' \to G$. We need to show that there is $f : A \to G$ extending f'.

We shall use Zorn's Lemma. Let $\Sigma = \{(B, g) | A' < B < A, g : B \to G, g|_{A'} = f'\}$. There exists a maximal element (M, h) in Σ. One verifies that $M = A$. □

proof of 4.4.5. $G \cong F/K$, $F \cong \sum_{x\in I} \mathbb{Z}x$. $F \xhookrightarrow{f} \sum_{x\in I} \mathbb{Q}x$. $G \cong F/K \cong f(F)/f(K) < \sum_{x\in I} \mathbb{Q}x/f(K)$ is divisible. □

Lemma 4.4.6. *Let $I^\bullet$ be an injective resolution of A and $J^\bullet$ an injective resolution of B. If there is $\varphi : A \to B$, then there exists $f : I^\bullet \to J^\bullet$ compatible with φ.*

Moreover any two such $f, g : I^\bullet \to J^\bullet$ are homotopic.

Definition 4.4.7. *$f, g \in \mathrm{Hom}(K^\bullet, L^\bullet)$ are homotopic if there are $h^i : K^i \to L^{i-1}$ such that $d_L h + h d_K = f - g$.*

Injective resolution is very useful in the study of left exact functors which is not exact. More, precise the following Lemma show that injective rsln splits

Lemma 4.4.8. *Given $0 \to A \to B \to C \to 0$, there is an exact sequence of complexes $0 \to I^\bullet \to J^\bullet \to K^\bullet \to 0$ such that $I^\bullet$ (resp, $J^\bullet, K^\bullet$) is an injective resolution of A (resp. B, C). Moreover, $J^i = I^i \oplus K^i$.*

Proof. We define I^0, K^0 first. Then there is a natural map $B \to J^0 := I^0 \oplus K^0$. This map is injective.

Then inductively, we get the resolutions. □

Warning: J is not $I \oplus K$ as complex. For example, the map $I^0 \oplus K^0 \to I^1 \oplus K^1$ is of the form $(d_I(i^0) + *, d_K(k^0))$ where $*$ is not necessarily zero.

We are now ready to study the left-exact functors. Apply F to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^\bullet & \longrightarrow & J^\bullet & \longrightarrow & K^\bullet & \longrightarrow & 0
\end{array}$$

We get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(A) & \longrightarrow & F(B) & \longrightarrow & F(C) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(I^\bullet) & \longrightarrow & F(J^\bullet) & \longrightarrow & F(K^\bullet) & \longrightarrow & 0
\end{array}$$

Notice that the bottom row is exact because $J^i = I^i \oplus K^i$ by our construction, hence $F(J^i) = F(I^i) \oplus F(K^i)$ for all i.

Proposition 4.4.9. *Let $R^iF(A) := H^i(F(I^\bullet))$. Then we have:*

(1) $R^0F(A) = A$.
(2) *there is a long exact sequence*

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to \ldots$$

Proof. It's easy to see that $\ker(F(I^0) \to F(I^1)) \cong F(A)$ by the left exactness. And the second statement follows from the long exact sequence of cohomology of short exact sequence of complexes. □

Exercise 4.4.10. *Show that $R^iF(A)$ is well-defined. That is, independent of choice of injective resolution.*

4.5. **derived category.** In this section, we are going to describe derived category briefly. It's a category over which cohomology theory can be defined and convenient to operate.

Exercise 4.5.1. *If h is homotopic to 0, denoted $h \sim 0$, then $fh \sim 0$, $hg \sim 0$ for all f, g whenever it makes sense.*

So we can think of the class of homotopic equivalence as an ideal.

Let $\mathcal{K}(\mathcal{A})$ be the category whose objects are complex in $\mathcal{A}$ and morphisms are morphism in $\mathcal{A}$ quotient homotopic equivalence. More precisely, $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(K^\bullet, L^\bullet)$ consists of homotopic equivalent class of $\mathrm{Hom}_{Kom(\mathcal{A})}(K^\bullet, L^\bullet)$.

Then in $\mathcal{K}(\mathcal{A})$, injective resolution is unique (up to isomorphism).

Definition 4.5.2. *Given a complex $K^\bullet = (K^i, d^i_K)$, we define $K[n]^\bullet$ such that $K[n]^i = K^{n+1}, d^i_{K[n]} = (-1)^n d^{n+i}_K$.*

And given a morphism $f : K^\bullet \to L^\bullet$, we define a complex $C(f)^\bullet$, called the **mapping cone of** *f, by $C(f)^i = K^{i+1} \oplus L^i$ and $d^i_C(k^{i+1}, l^i) = (-d^{i+1}_K(k^{i+1}), f(k^{i+1}) + d^i_L(l^i))$.*

Example 4.5.3.

If $K^\bullet = K, L^\bullet = L$, then $C(f) = 0 \to K \to L \to 0$. □

Example 4.5.4.

If $f = 0$, then $C(f) = K^\bullet \oplus L^\bullet$.

Definition 4.5.5. *Given a morphsim $f : K^\bullet \to L^\bullet$, we define a complex $Cyl(f)^\bullet$ such that $Cyl(f)^i = K^i \oplus K^{i+1} \oplus L^i$. And $d^i_{Cyl}(k^i, k^{i+1}, l^i) = (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L l^i)$.*

Theorem 4.5.6. *We have the following diagram*

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & L^\bullet & \longrightarrow & C(f)^\bullet & \xrightarrow{\delta} & K[1]^\bullet & \longrightarrow & 0 \\
 & & & & \alpha\downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K^\bullet & \xrightarrow{\bar{f}} & Cyl(f)^\bullet & \xrightarrow{\pi} & C(f)^\bullet & \longrightarrow & 0 & & \\
 & & =\downarrow & & \beta\downarrow & & & & & & \\
 & & K^\bullet & \xrightarrow{f} & L^\bullet & & & & & &
\end{array}$$

Such that each row is exact. α, β *are quasi-isomorphisms. Moreover,* $\beta\alpha = \mathbf{1}_L$ *and* $\alpha\beta \sim \mathbf{1}_{Cyl(f)}$.

Proof. All the above maps are the natural ones. One has to check that all the maps indeed gives morphism of complexes and the diagram commutes. We leave the detail to the readers.

The homotopy is defined by $h^i(k^i, k^{i+1}, l^i) = (0, k^i, 0)$. □

Theorem 4.5.7. *Given an exact sequence* $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$*, we have the following commutative diagram with each vertical map being quasi-isomorphic.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^\bullet & \xrightarrow{f} & L^\bullet & \xrightarrow{g} & M^\bullet & \longrightarrow & 0 \\
 & & \uparrow & & \beta\uparrow & & \gamma\uparrow & & \\
0 & \longrightarrow & K^\bullet & \xrightarrow{\bar{f}} & Cyl(f)^\bullet & \xrightarrow{\pi} & C(f)^\bullet & \longrightarrow & 0
\end{array},$$

where $\gamma(k^{i+1}, l^i) = g(l^i)$.

The second row is called **distinguished triangle**.

Derived category $D(\mathcal{A})$ is the category localizing $\mathcal{K}(\mathcal{A})$ with respect to quasi-isomorphisms. That is, a morphism $\mathrm{Hom}_{D(\mathcal{A})}(X, Y)$ in $D(\mathcal{A})$ is a roof (t, f) where $t : Z^\bullet \to X^\bullet$ is a quasi-isomorphism and $f : Z^\bullet \to Y^\bullet$ is a morphism in $\mathcal{K}(\mathcal{A})$. Then in this setting, a quasi-isomorphism $s : X^\bullet \to Y^\bullet$ has inverse $(s, \mathbf{1}_X) \in \mathrm{Hom}_{D(\mathcal{A})}(Y^\bullet, X^\bullet)$.

Derived category has the universal property that any functor $F : Kom(\mathcal{A}) \to \mathcal{D}$ sending quasi-isomorphism into isomorphism can be uniquely factored through $D(\mathcal{A})$.

Note that a cohomology (homology) theory on $\mathcal{A}$ is nothing but a functor $F : Kom(\mathcal{A}) \to Kom(Ab)$ and thus factors through derived category.