Jan. 5, 2007

4.3. **complexes, exact sequences.** .

Definition 4.3.1. *By a short exact sequence, we mean an exact sequence* $0 \to A \to B \to C \to 0$.

Example 4.3.2.

1. Let A, B be abelian groups, then we have exact sequence:
$$0 \to A \xrightarrow{\imath_A} A \oplus B \xrightarrow{p_B} B \to 0.$$
2. Let $A \lhd B$ be abelain groups, then we have exact sequence:
$$0 \to A \to B \to B/A \to 0.$$
3. Let $\varphi : B \to C$ be a surjective homomorphism, then we have exact sequence:
$$0 \to \ker(\varphi) \to B \to C \to 0.$$

$\square$

Given a long exact sequence $K^\bullet = (K^i, d_i)$, it can be decomposed into short exact sequences
$$0 \to \ker(d^i) = \mathrm{im}(d^{i-1}) \to K^i \to \mathrm{im}(d^i) = \ker(d^{i+1}) \to 0.$$
Therefore, short exact sequences play the most important role in our studies.

Given a morphism $\phi \in \mathrm{Hom}(K^\bullet, L^\bullet)$ of complexes, one can define its kernel, image, cokernel, in a natural way. Thus we can formulate a new category $Kom(\mathcal{A})$, whose objects are complexes over $\mathcal{A}$ and morphisms are morphism of complexes.

Exercise 4.3.3. *$Kom(\mathcal{A})$ is an abelian category in which $\mathcal{A}$ is a subcategory.*

Let $K^\bullet$ be a complex. We let $Z^i := \ker(d^i)$, called the i**-th cocycle** and $B^i := \mathrm{im}(d^{i-1})$, called the i**-th coboundary**. Then $H^i(K^\bullet) := Z^i/B^i$ is called the i**-th cohomology** of $K^\bullet$. Cohomology can be viewed as a tool detecting the non-exactness of complexes.

Given two complexes $K^\bullet, L^\bullet$, a morphism of complexes $\phi \in Hom_{\mathcal{A}}(K^\bullet, L^\bullet)$ consists of morphisms $\phi^i : K^i \to L^i$ such that $\phi^{i+1} \circ d^i_K = d^i_L \circ \phi^i$ for all i. Another way to put it is the following diagram commute:

$$\begin{tikzcd}
\arrow[r] & K^i \arrow[r, "d^i_K"] \arrow[d, "\phi^i"'] & K^{i+1} \arrow[r] \arrow[d, "\phi^{i+1}"'] & {} \\
\arrow[r] & L^i \arrow[r, "d^i_L"] & L^{i+1} \arrow[r] & {}
\end{tikzcd}$$

One can easily checked that there is an induced map $H^i(\phi) : H^i(K^\bullet) \to H^i(L^\bullet)$ for all i. Moreover, if ϕ, ψ are morphism of complexes, then $H^i(\psi) \circ H^i(\phi) = H^i(\psi \circ \phi)$ for all i whenever it make sense.

Before we move on, we discuss the following useful lemmas:

Lemma 4.3.4 (Snake Lemma). *Given a diagram*

$$\begin{array}{ccccccc} & & A' & \xrightarrow{f} & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & {\scriptstyle d'}\big\downarrow & & {\scriptstyle d}\big\downarrow & & {\scriptstyle d''}\big\downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \xrightarrow{g} & B'' & & \end{array}$$

with each rows are exact. Then there is a well-defined map $\delta : \ker(d'') \to \operatorname{coker}(d')$ *such that we have an exact sequence*

$$\ker(d') \xrightarrow{f} \ker(d) \to \ker(d'') \xrightarrow{\delta} \operatorname{coker}(d') \to \operatorname{coker}(d) \xrightarrow{\bar{g}} \operatorname{coker}(d'').$$

If moreover that $f : A' \to A$ *is injective, then* $f : \ker(d') \to \ker(d)$ *is injective. And if* $g : B \to B''$ *is surjective, then* $\bar{g} : \operatorname{coker}(d) \to \operatorname{coker}(d'')$ *is surjective.*

Proof. The proof consists of various diagram chasing. We leave it to the reader. □

Corollary 4.3.5. *Keep the notation as above. If both* d', d'' *are injective (resp. surjective) then so is* d.

Assume that f *is injective and* g *is surjective. If any two of* d', d, d'' *are isomorphism. So is the third one.*

Lemma 4.3.6 (Five Lemma). *Given a diagram*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ {\scriptstyle d_1}\big\downarrow & & {\scriptstyle d_2}\big\downarrow & & {\scriptstyle d_3}\big\downarrow & & {\scriptstyle d_4}\big\downarrow & & {\scriptstyle d_5}\big\downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & b_5 \end{array}$$

with each rows are exact.

If d_1 *is surjective (resp. injective) and* d_2, d_4 *are injective (resp. surjective), then* d_3 *is injective (resp. surjective).*

In particular, if d_1, d_2, d_4, d_5 *are isomorphic, then so is* d_3.

Proof. Decompose the sequence into short exact sequences. □

An immediate application is the following:

Proposition 4.3.7. *Given an exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, *the following are equivalent:*

1. *there is* $h : C \to B$ *such that* $gh = \mathbf{1}_C$.
2. *there is* $l : B \to A$ *such that* $lf = \mathbf{1}_A$.
3. *the sequence is isomorphic to* $0 \to A \xrightarrow{\imath_A} A \oplus C \xrightarrow{p_C} C \to 0$.

Such sequence is called **split**.

If the sequence split, then in particular, $B \cong A \oplus C$.

Proof. Given $h : C \to B$, we can construct the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\imath_A} & A \oplus C & \xrightarrow{p_C} & C & \longrightarrow & 0 \\
\downarrow & & {\scriptstyle \mathbf{1}_A}\downarrow & & {\scriptstyle fp_A + hp_C}\downarrow & & {\scriptstyle \mathbf{1}_C}\downarrow & & \downarrow \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

By Five Lemma, $fp_A + hp_C$ is an isomorphism. Hence those two sequences are isomorphic.

On the other hand, if the two sequence are isomorphic. That is we have the following commutative diagram, which is invertible:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\imath_A} & A \oplus C & \xrightarrow{p_C} & C & \longrightarrow & 0 \\
 & & {\scriptstyle \mathbf{1}_A}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle \mathbf{1}_C}\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

Let $h = \phi \circ \imath_C : C \to B$, then $gh = g\phi\imath_C = \mathbf{1}_C p_C \imath_C = \mathbf{1}_C$.

The proof for other equivalence is similar. $\square$

Theorem 4.3.8. *Given a short exact of complexes, then it induces a long exact sequences of cohomology.*

Proof. This can be proved directly, or by Snake Lemma.

We briefly sketch the proof by using Snake Lemma here.

First look at the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{i-1} & \longrightarrow & B^{i-1} & \longrightarrow & C^{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{i} & \longrightarrow & B^{i} & \longrightarrow & C^{i} & \longrightarrow & 0
\end{array}$$

Then we have exact sequence $A^i/B^i(A^\bullet) \to B^i/B^i(B^\bullet) \to C^i/B^i(C^\bullet) \to 0$ by looking at cokernel of the maps.

Next we look at the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{i+1} & \longrightarrow & B^{i+1} & \longrightarrow & C^{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{i+2} & \longrightarrow & B^{i+2} & \longrightarrow & C^{i+2} & \longrightarrow & 0
\end{array}$$

Then we have exact sequence $0 \to Z^{i+1}(A^\bullet) \to Z^{i+1}(B^\bullet) \to Z^{i+1}(C^\bullet)$ by looking at kernels.

These two exact sequences fit into a commutative diagram

$$\begin{array}{ccccccccc}
 & & A^i/B^i(A^\bullet) & \longrightarrow & B^i/B^i(B^\bullet) & \longrightarrow & C^i/B^i(C^\bullet) & \longrightarrow & 0 \\
 & & {\scriptstyle \bar{d}^i_A}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^{i+1}(A^\bullet) & \longrightarrow & Z^{i+1}(B^\bullet) & \longrightarrow & Z^{i+1}(C^\bullet) & &
\end{array}$$

One can check that $\ker(\bar{d}^i_A) = H^i(A^\bullet)$ and $\operatorname{coker}(\bar{d}^i_A) = H^{i+1}(A^\bullet)$. And similarly for $B^\bullet$ and $C^\bullet$. Hence by Snake Lemma, we are done. $\square$

Definition 4.3.9. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between two abelian categories. We say that F is* **exact** *if for an exact sequence $K^\bullet$ over over $\mathcal{A}$, $F(K^\bullet)$ is exact over $\mathcal{B}$.*

Exercise 4.3.10. *Show that F is exact if and only if for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the induced sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact in $\mathcal{B}$.*

Definition 4.3.11. *Keep the notation as above. We say that F is left-exact (resp. right-exact) if for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the induced sequence $0 \to F(A) \to F(B) \to F(C)$ (resp. $F(A) \to F(B) \to F(C) \to 0$) is exact in $\mathcal{B}$.*

Unfortunately, most natural functors are left-exact (or right-exact) but not exact. We list some of them:

Example 4.3.12.

Let X be a topological space. Let Sh_X be the category of sheaves on X, which is an abelian category. The global section functor $\Gamma(X, \cdot) : Sh_X \to Ab$ is left exact but not exact. $\square$

Example 4.3.13.

Let Ab be the category of abelian groups. Fixed $M \in Ab$, we consider $\mathrm{Hom}(M.\cdot) : Ab \to Ab$ by $A \mapsto \mathrm{Hom}(M, A)$. This is left-exact but nor right exact. $\square$