Advanced Algebra I

Homework 9

due on Dec. 8, 2006

(1) Complete the uncompleted proof in the lecture.

(2) Let $k[u_1, ..., u_n]$ be a polynomial ring with n indeterminates. Let $s_i := \sum_{j_1<j_2<K<j_i} u_{j_1}u_{j_2}...u_{j_i}$ be the elementary symmetric polynomials. Show that $f \in k[u_1, ..., u_n]$ is symmetric if and only if $f \in k[s_1, ..., s_n]$.

In fact, one can consider the group action S_n on $k[u_1, ..., u_n]$. Let $k[u_1, ..., u_n]^{S_n}$ be the **invariant**, i.e. $\{f|\sigma(f) = f, \forall\sigma \in S_n\}$. The above assertion can be rephrased as $k[u_1, ..., u_n]^{S_n} = k[s_1, ..., s_n]$.

One can show that $k(u_1, ..., u_n)^{S_n} = k(s_1, ..., s_n)$.

(3) Keep the notation as above. Determine the Galois group of $k(u_1, ..., u_n)$ over $k(s_1, ..., s_n)$.

(4) Show that for any given finite group G. There exists a Galois extension F/K with Galois group G.
(Remark: It would be a very difficult problem (Inverse Galois Problem) if the base field is fixed, e.g $\mathbb{Q}$.)

(5) Determine the Galois group of $x^3 + x + 1$ over $\mathbb{Q}$ and over $\mathbb{F}_5$ respectively.

(6) Determine the Galois group of $x^4 + x + 1$ over $\mathbb{Q}$ and over $\mathbb{F}_7$ respectively.

(7) Determine the Galois group of $x^7 - 3$ over $\mathbb{Q}$.

(8) If $f(x) = x^4 + bx^3 + cx^2 + dx + e$, then its resolvant cubic is $g(x) = x^3 - cx^2 + (bd - 4e)x - b^2e + 4ce - d^2$.