Advanced Algebra I

Homework 14

due on Jan. 12, 2007

(1) Complete the uncompleted proof in the lecture. Especially, the proof of Snake Lemma and Five Lemma.

(2) Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}$. By a **product** of A, B we mean a triple (P, f, g) that consists of $P \in \mathcal{C}$ and $f : P \to A, g : P \to B$ such that for all (C, α, β) with $\alpha : C \to A, \beta : C \to B$, there exists a unique morphism $h : C \to P$ such that $fh = \alpha, gh = \beta$.

  (a) Formulate a category so that (P, f, g) corresponds to a universal object in that category.

  (b) Let $Z \in \mathcal{C}$. Then we can have a new category $\mathcal{C}_Z$, whose objects are (X, f) with $f : X \to Z$ in $\mathcal{C}$. And morphisms in $\mathrm{Hom}_{\mathcal{C}_Z}((X, f), (Y, g))$ are morphism $h : X \to Y$ such that $gh = f$. Describe the product of $(X, f), (Y, g)$. (This is called the **fibered product of X, Y over Z**.)

  (c) By a coproduct of A, B we mean a triple (C, f, g) consists of $C \in \mathcal{C}$ and $f : A \to C, g : B \to C$ such that for all (D, α, β) with $\alpha : A \to D, \beta : B \to D$, there exists a unique morphism $h : C \to D$ such that $hf = \alpha, hg = \beta$.

  (d) Define and describe the fiber coproduct.

  (e) Show that the product and coproduct exist in the category of sets.

  (f) Show that fibered product and fibered coproduct exist in the category of abelian groups.

(3) Let $\mathcal{A}$ be an abelian category. Show that $Kom(\mathcal{A})$ is again an abelian category.

(4) Here is the 3×3 Lemma. Given following commutative diagram in an abelian category such that each column is exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

  (a) If the bottom two rows are exact, so is the top row.

(b) If the top two rows are exact, so is the bottom row.

(c) If the top and bottom rows are exact and the composition $A \to C$ is zero, then the middle row is exact.

(5) Given a complex $K^\bullet$. For a fixed n, we would like to have a new complex $L^\bullet$ such that

$$L^i = \begin{cases} 0 & \text{if } i < n \\ K^i & \text{if } i > n. \end{cases}$$

And moreover $H^i(L^\bullet) = H^i(K^\bullet)$ for $i \geq n$ and $H^i(L^\bullet) = 0$ for $i < n$. How to define L^n?

If similarly we would like to have a complex $M^\bullet$ such that $H^i(M^\bullet) = 0$ for all $i > n$ and $H^i(M^\bullet) = H^i(K^\bullet)$ for $i \leq n$. What can you do?