Advanced Algebra I
Homework 12
due on Dec. 29, 2006

(1) Complete the uncompleted proof in the lecture. Especially, the
properties of purely inseparable extension.

(2) Consider the extension $F := \mathbb{Q}[\sqrt{2}, \sqrt{3}]/\mathbb{Q}$. Find an element
$u \in F$ such that $F = \mathbb{Q}[u]$ and verify your answer.

(3) Construct an example of algebraic extension $F/K$ such that $F$
is not separable over $P$.

(4) Show that the separable extension has the following properties:
(a) Let $K \subset E \subset F$. Then $F/K$ is separable if and only if
$F/E$ and $E/K$ are separable.
(b) If $E/K$ is separable then $FE/F$ is separable for an exten-
sion $F/K$.
(c) If $E, F \subset L$ are separable extension over $K$. Then $EF$ is
separable over $K$.

How about purely inseparable extension?

(5) Let $\text{char} K = p$. If $a \in K - K^p$, then $x^{p^n} - a$ is irreducible for
every $n \geq 1$.

(6) Consider $K(x)/K$ the field of one indeterminate over $K$. Show
that an intermediate subfield is of the form $K(t)$ for some $t \in
K(x)$.

(7) Consider the extension $K(x, y)/K$.
(a) Verify that there is a group action $PGL(3, K) \times K(x, y) \to
K(x, y)$ by $g(x, y) := \left(\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right)$ if $g =
\left(\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right)$. Hence it induces a group homomor-
phism $\varphi : PGL(3, K) \to \text{Aut}(K(x, y))$. Also show that $\varphi$
is injective.
(b) Can you find some other automorphisms in $\text{Aut}(K(x, y)) -
PGL(3, K)$?
(c) * Describe $\text{Aut}(K(x, y))$. 