Advanced Algebra I

Homework 12

due on Dec. 29, 2006

(1) Complete the uncompleted proof in the lecture. Especially, the properties of purely inseparable extension.

(2) Consider the extension $F := \mathbb{Q}[\sqrt{2}, \sqrt{3}]/\mathbb{Q}$. Find an element $u \in F$ such that $F = \mathbb{Q}[u]$ and verify your answer.

(3) Construct an example of algebraic extension F/K such that F is not separable over P.

(4) Show that the separable extension has the following properties:

(a) Let $K \subset E \subset F$. Then F/K is separable if and only if F/E and E/K are separable.

(b) If E/K is separable then FE/F is separable for an extension F/K.

(c) If $E, F \subset L$ are separable extension over K. Then EF is separable over K.

How about purely inseparable extension?

(5) Let $\mathrm{char} K = p$. If $a \in K - K^p$, then $x^{p^n} - a$ is irreducible for every $n \geq 1$.

(6) Consider $K(x)/K$ the field of one indeterminate over K. Show that an intermediate subfield is of the form $K(t)$ for some $t \in K(x)$.

(7) Consider the extension $K(x, y)/K$.

(a) Verify that there is a group action $PGL(3, K) \times K(x, y) \to K(x, y)$ by $g(x, y) := (\frac{a_{11}x+a_{12}y+a_{13}}{a_{31}x+a_{32}y+a_{33}}, \frac{a_{21}x+a_{22}y+a_{23}}{a_{31}x+a_{32}y+a_{33}})$ if $g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Hence it induces a group homomorphism $\varphi : PGL(3, K) \to \mathrm{Aut}(K(x, y))$. Also show that φ is injective.

(b) Can you find some other automorphisms in $\mathrm{Aut}(K(x, y)) - PGL(3, K)$?

(c) * Describe $\mathrm{Aut}(K(x, y))$.