Advanced Algebra I

Homework 11

due on Dec. 22, 2006

(1) Complete the uncompleted proof in the lecture. Especially, the properties of Lagrange resovents.

(2) Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree p, where p is an odd prime. Suppose furthermore that $f(x)$ has only 2 complex roots. Then Galois group of $f(x)$ is S_p.

(3) (*) Construct an example $f(x) \in \mathbb{Q}[x]$ such that Galois group is A_5.

(4) Let p be prime. We consider $f_{a,b} : \mathbb{Z}_p \to \mathbb{Z}_p$, given by $f_{a,b}(x) = ax + b$ with $a \neq 0$. Let

$$GA(p) := \{f_{a,b} | a, b \in \mathbb{Z}_p, a \neq 0\}.$$

  (a) $GA(p)$ can be viewed as a subgroup of S_p. It is solvable and transitive of order $|GA(p)| = p(p-1)$.

  (b) (*) A subgroup $H < S_p$ which is solvable and transitive is conjugate to a subgroup of $GA(p)$.

(5) Consider $f(x) = x^p - x - a \in \mathbb{F}_p[x]$. Is it irreducible for all $a \in \mathbb{F}_p^*$?

(6) For any give cyclic group $\mathbb{Z}_n$, there exists a finite Galois extension $F/\mathbb{Q}$ whose Galois group is $\mathbb{Z}_n$.
(*) For For any give abelain group G, there exists a finite Galois extension $F/\mathbb{Q}$ whose Galois group is G.

(7) Give an example of $F/\mathbb{Q}$ such that F is contained in a radical extension, but $F/\mathbb{Q}$ is not radical.