Advanced Algebra I

Homework 10

due on Dec. 15, 2006

(1) * Complete the uncompleted proof in the lecture.

(2) Consider the cyclotomic extension $\mathbb{Q}(\zeta)/\mathbb{Q}$, where $\zeta = e^{\frac{2\pi i}{7}}$. Determine its Galois groups, all intermediate subfields, and the minimal polynomial of $\zeta + \zeta^{-1}$ over $\mathbb{Q}$.

(3) Every element in a finite field can be written as sum of two squares.

(4) Let F be the algebraic closure of $\mathbb{F}_p$.
  (a) The frobenius map $\varphi : u \mapsto u^p$ is a $\mathbb{F}_p$-automorphism.
  (b) F is Galois over $\mathbb{F}_p$.
  (c) The subgroup generated by φ is a proper subgroup.
  (d) $\mathrm{Gal}_{\mathbb{F}_p} F$ is abelian.

(5) (*) What can you say on the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$?

(6) We consider cyclotomic polynomials over $\mathbb{Q}$.
  (a) If p is prime, then $g_{p^k}(x) = g_p(x^{p^{k-1}})$.
  (b) If p is prime and $(p, n) = 1$, then $g_{pn}(x) = \frac{g_n(x^p)}{g_n(x)}$.
  (c) $g_n(1) = \begin{cases} p & \text{if } n = p^k (k > 0), \\ 0 & \text{if } n = 1, \\ 1 & \text{otherwise.} \end{cases}$

(7) How many irreducible polynomials of degree n over $\mathbb{F}_p$?