Introduction to Algebraic Geometry

Homework 1

Discuss on Oct. 7, 2005. Due on Oct. 14, 2005

In all homeworks, k denotes an algebraically closed field unless otherwise stated.

(1) Read the handout on localization.

(2) Let R be a commutative ring with identity, $I \lhd R$ be an ideal, $\mathfrak{p} \lhd R$ be a prime ideal, and $f \in R$. Describe the correspondence between prime ideals and primes ideals of R/I, $R_{\mathfrak{p}}$, R_f.

(3) An algebraically closed field is infinite. Let $f \in k[x_1, ..., x_n]$ be a non-constant polynomial with $n \geq 2$, show that $\mathcal{V}(f)$ is infinite. (Hint: by induction on number of variables).

(4) Consider the morphism $\phi_1 : \mathbb{P}^1 \to \mathbb{P}^3$ by $\phi_1(s : t) \mapsto (s^3, s^2t, st^2, t^3)$. Show that $\phi_1(\mathbb{P}^1) = \mathcal{V}(Z_0Z_2 - Z_1^2, Z_0Z_3 - Z_1Z_2, Z_1Z_3 - Z_2^2)$. Is $k[Z_0, Z_1, Z_2, Z_3]/(Z_0Z_2 - Z_1^2, Z_0Z_3 - Z_1Z_2, Z_1Z_3 - Z_2^2) \cong k[X_0, X_1]$?

If we consider the morphism $\phi_2 : \mathbb{P}^1 \to \mathbb{P}^3$ by $\phi_2(s : t) \mapsto (s^4, s^3t, st^3, t^4)$, what is $\mathcal{I}(\phi_2(\mathbb{P}^1))$?

(5) Let k be a field of $char(k) = p > 0$. We consider $Fr : \mathbb{A}^1 \to \mathbb{A}^1$ by $Fr(s) \mapsto s^p$. Show that Fr is bijective but not an isomorphism. What are the fixed points of Fr?

(6) Consider $F_1 = XY - Z^2, F_2 = X^2 + Y^2 - Z^2$.

(a) Count the number of points of $\mathcal{V}(F_1), \mathcal{V}(F_2)$ in $\mathbb{A}^3_{\mathbb{F}_7}$, with $\mathbb{F}_7$ denotes the field of 7 elements.

(b) Show that $\mathcal{V}(F_1)$ and $\mathcal{V}(F_2)$ are isomorphic in $\mathbb{A}^3_{\overline{\mathbb{F}}_7}$, with $\overline{\mathbb{F}}_7$ denotes the algebraically closure of $\mathbb{F}_7$.