Introduction to Algebraic Geometry

Sep. 14, 2005 (Fri.)

1. Geometry of Varieties

1.1. **Local functions on varieties.** In order to consider function locally at $p \in X$, the strategy is considering *germs of function*, i.e. (U, f), where U is an open neighborhood of x and $f : U \to k$ is a regular function on U. By regular functions, we mean:

Definition 1.1.1. *Let $X \subset \mathbb{A}^n$ be an affine variety and $U \subset X$ is an open set. We say that f is a regular function on U, if $f = \frac{h}{g}$ with $g, h \in k[x_1, ..., x_n]$ such that $g(p) \neq 0$ for all $p \in U$.*

However, if $(U, f), (V, g)$ are two germs near p that $f|_{U \cap V} = g|_{U \cap V}$, they should be treated as the same function near p. Therefore, we would like to introduce an equivalent relation naively that

$$(U, f) \sim (V, g) \iff f|_{U \cap V} = g|_{U \cap V}.$$

However, one might have difficulty to check the transitivity. It turns out that a better definition of the required equivalent is

$$(U, f) \sim (V, g) \iff \exists W \subset U \cap V, f|_W = g|_W.$$

Let $\mathcal{O}_{p,X}$ be the equivalent classes of germs of regular function near p. It's easy to check that $\mathcal{O}_{p,X}$ has a natural ring structure. And in fact, it's a local ring with the maximal ideal can be describe as

$$\mathfrak{m}_p := \{(U, f) | f(p) = 0\} / \sim .$$

Proposition 1.1.2. *Let X be an affine variety with coordinate ring $\Gamma(X)$, and $\mathfrak{n}_p$ is the maximal ideal of $\Gamma(X)$ corresponding to the point p. Then there is an natural isomorphism $\Gamma(X)_{\mathfrak{n}_p} \to \mathcal{O}_{p,X}$.*

Recall that $\Gamma(X)$ can be viewed as "polynomial functions on $X \subset \mathbb{A}^n$". We now can reinterpret it in a more geometrical form. We say a function is regular on X if it's regular at every point of X. Let $\mathcal{O}(X)$ be the ring (it's clearly a ring) of regular function on X. Of course, we have

$$\mathcal{O}(X) = \cap_{p \in X} \mathcal{O}_{p,X}.$$

In fact we the the following:

Proposition 1.1.3. *Let X be an affine variety. Then there is a natural isomorphism $\Gamma(X) \to \mathcal{O}(X)$.*

The next step is to study regular function on open set and closed set of X. The closed sets of X are again an affine variety corresponding to a prime ideal $\mathfrak{p} \lhd \Gamma(X)$. So if $Y = \mathcal{V}(\mathfrak{p}) \subset X$, then

$$\mathcal{O}(Y) = \Gamma(X)/\mathfrak{p} = \Gamma(Y).$$

The general open subset of X might be far from reach. It's easy to handle the open subsets of the form $X_f := X - \mathcal{V}(f)$ for $f \neq 0 \in \Gamma(X)$. Recall that such open subsets form a basis of the Zariski topology of X. So we imagine that we didn't lose anything by restricting ourselves to this collection. Then we leave it as an exercise to show that

$$\Gamma(X)_f \cong \mathcal{O}(X_f).$$

Lastly, we introduce the function field, denoted $K(X) := \{(U, f) | U \neq \emptyset \subset X, f : U \to k \text{ is regular}\} / \sim$, to be the equivalent classless of germs of function on X. Notice that any two non-empty open subset of X must intersect because X is irreducible. So it's possible to define ring structure on $K(X)$ naturally. We leave it an exercise to show that $K(X)$ is nothing but the quotient field of $\Gamma(X)$.

1.2. **Morphisms.**

Definition 1.2.1. *Let X, Y be varieties (projective or affine). A continuous map $\varphi : X \to Y$ is said to be a morphism if for $(U, f) \in K(Y)$ with $U \cap \varphi(X) \neq \emptyset$, then $(\varphi^{-1}(U), f \circ \varphi) \in K(X)$.*

Exercise 1.2.2. *If both X, Y are affine, then φ is a morphism if and only if it's a polynomial map.*

Example 1.2.3. *Consider the map $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ by $\varphi(s : t) \mapsto (s^2 : st : t^2)$. Then it's a morphism.*

In fact, one can show that a morphism between projective varieties is nothing but homogeneous polynomial maps.

1.3. **Dimension theory.** The dimension is a basic notion in geometry. The purpose of this section is to define dimension of varieties algebraically and derive some properties which meet our geometrical intuition.

We define

Definition 1.3.1. *Let X be an variety. Then we define*

$$\dim X := \text{tr.d.}_{\text{k}k} K(X).$$

The transcendental degree is the maximal number of algebraically independent elements in $K(X)$. One can think it as maximal number "parameter" on X which meets our intuition of dimension.

Example 1.3.2. *Let $X = \mathcal{V}(y^2 + x^2 - 1) \subset \mathbb{A}^2$. Near $(1, 0)$, one can use y as the parameter and near $(0, 1)$ one can use x as parameter. It's easy to see that the maximal number of parameter is 1.*

We will see that the transcendental degree of $K(X)$ is 1 as well.

Before we go on, we observe that $K(X) = K(U)$ for any non-empty open $U \subset X$. Thus, if $X \subset \mathbb{P}^n$ is a projective variety, then $X \cap U_i \neq \emptyset$ for some $i = 0, ..., n$, say U_0. Then $K(X) = K(X \cap U_0)$. Nevertheless,

$X \cap U_0$ can be identified with an affine variety via the natural morphism $\varphi_0 : U_0 \to \mathbb{A}^n$. We thus may restrict our discussion to affine varieties.

Proposition 1.3.3. *Let $Y \subset X$ be a proper subvariety. Then* $\dim Y < \dim X$*.*

This is euqivalent to the following algebraic version

Proposition 1.3.4. *Let R be an finitely generated integral domain over k. Let $\mathfrak{p} \lhd R$ be a prime ideal. Then*

$$tr.d._k R \geq tr.d._k R/\mathfrak{p}$$

with equality holds only when $\mathfrak{p} = 0$.

Note that tr.d.$_k R$ is nothing but tr.d.$_k F$ with F the quotient field of R.

Proof. Suppose that $\mathfrak{p} \neq 0$ and let $n :=$ tr.d.$_k R$. Suppose on the contrary that tr.d.$_k R.\mathfrak{p} = n$, there are n elements in R, say $s_1, ..., s_n$ such that their image, say $\bar{s}_i$, are algebraically independent in $R/\mathfrak{p}$.

Pick any $t \neq 0 \in \mathfrak{p}$, then $\{s_1, ..., s_n, t\}$ must be algebraically dependent. So they satisfies a polynomial $F(x_1, ..., x_n, y)$ over k. We may assume that F is irreducible.

It's easy to see that $y \not| F$. So expand F w.r.t. to y, we write

$$F = F_0(x_1, ..., x_n) + F_1(x_1, ..., x_n)y + ...$$

Passing $0 = F(s_1, ..., s_n, t)$ to $R/\mathfrak{p}$, one sees that $F_0(\bar{s}_1, ..., \bar{s}_n) = 0$. This leads to the required contradiction. □

This Proposition also suggest that the Zariski topology is pretty coarse, i.e. there are not so many closed set in there. We can also imagine that the largest subvariety of X has dimension $\dim X - 1$. This leads to topological characterization of dimension of varieties by

Definition 1.3.5. *Let X be an affine variety. Then we define*

$$\dim' X := sup\{n | Y_0 \subsetneq Y_1 \subsetneq ... \subsetneq Y_n = X\}.$$

If we translate this into algebra. Then $\dim' X$ is nothing but the Krull dimension of $\Gamma(X)$, denoted $\dim \Gamma(X)$.

To justify our definition, we need the following:

Theorem 1.3.6. *Let R be a finitely generated domain over k. Then* $\dim R = tr.d._k R$*. Therefore, for an affine variety X,* $\dim X = \dim' X$*.*

Theorem 1.3.7 (Krull's Haupt-ideal-satz=principal ideal theorem)**.** *Let R be a finitely generated domain over k. Let $f \in R$ be an element which is neither zero nor a unit. Then every minimal prime ideal $\mathfrak{p}$ containing f has height 1.*

proof of Theorem 1.3.6. We first claim that $\dim R \leq \text{tr.d.}_\text{k} R$. To see this, it suffices to show that if $\mathfrak{p} \subsetneq \mathfrak{q} \in \text{Spec}(R)$, then $\text{tr.d.}_\text{k} R/\mathfrak{p} \gneq \text{tr.d.}_\text{k} R/\mathfrak{q}$.

Let $\{\beta_1, ..., \beta_r\}$ be a transcendental basis of $R/\mathfrak{q}$. Then it lifts to $\{\alpha_1, ..., \alpha_r\}$ which is algebraically independent in $R/\mathfrak{p}$. This is because there is a surjective homomorphism $\varphi : R/\mathfrak{p} \to R/\mathfrak{q}$. If there is an algebraic relation among $\{\alpha_1, ..., \alpha_r\}$ then it gives an relation among $\{\beta_1, ..., \beta_r\}$ via the homomorphism φ, which is absurd. Hence we have shown that $\text{tr.d.}_\text{k} R/\mathfrak{p} \geq \text{tr.d.}_\text{k} R/\mathfrak{q}$.

Assume now that $\text{tr.d.}_\text{k} R/\mathfrak{p} = \text{tr.d.}_\text{k} R/\mathfrak{q}$. Then $\{\alpha_1, ..., \alpha_r\}$ is an basis. Lift to R, we have an algebraically independent set $\{y_1, ..., y_r\} \subset R$. Let $S = k[y_1, ..., y_r] - \{0\}$. We are going to localize w.r.t S. Write $R/\mathfrak{p}$ as $k[\alpha_1, ..., \alpha_r, \alpha_{r+1}, ..., \alpha_n]$, then

$$S^{-1}R/S^{-1}\mathfrak{p} = \bar{S}^{-1}(R/\mathfrak{p}) = k(\alpha_1, ..., \alpha_r)[\alpha_{r+1}, ..., \alpha_n].$$

We claim that $S^{-1}R/S^{-1}\mathfrak{p}$ is a field, hence $S^{-1}\mathfrak{p}$ is a maximal ideal. However, note that $S \cap \mathfrak{p} = S \cap \mathfrak{q} = \emptyset$. It follows that $S^{-1}\mathfrak{p}$ and $S^{-1}\mathfrak{q}$ are prime ideals of $S^{-1}R$.

$$S^{-1}\mathfrak{p} \subsetneq S^{-1}\mathfrak{q} \subsetneq S^{-1}R.$$

This is the required contradiction.

To see the claim, let $F = k(\alpha_1, ..., \alpha_n)$ be the quotient field of $R/\mathfrak{p}$. Since $\{\alpha_1, ..., \alpha_r\}$ is a transcendental basis, we have that α_j is algebraic over $K := k(\alpha_1, ..., \alpha_r)$ for all $r < j \leq n$. For algebraic element, it's clear that ring extension is a field extension. Thus $F = K(\alpha_{r+1}, ..., \alpha_n) = K[\alpha_{r+1}, ..., \alpha_n]$. The claim follows.

The next step is to show that $\dim R \geq \text{tr.d.}_\text{k} R$. This follows from Noether normalization lemma, that is, reduce to polynomial ring. More precisely, let $r := \text{tr.d.}_\text{k} R$, then there exist algebraic independent $\{y_1, ..., y_r\} \subset R$ and R is integral over $k[y_1, ..., y_r]$. We have $\dim R = \dim k[y_1, ..., y_r]$ since it's an integral extension. And also, $\dim k[y_1, ..., y_r] \geq r$ because clearly we have a chain of prime ideals of length n,

$$0 \subset (y_1) \subset (y_1, y_2) ... \subset (y_1, ..., y_r).$$

This completes the proof. □

proof of Theorem 1.3.7. Let $\mathfrak{p}$ be a minimal prime containing f. We will show that $\text{tr.d.}_\text{k} R/\mathfrak{p} = \text{tr.d.}_\text{k} R - 1$, where $R = \Gamma(X)$.

Suppose that $\sqrt{(f)} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap ... \cap \mathfrak{p}_t$ with $\mathfrak{p} = \mathfrak{p}_1$. We can pick $g \in \mathfrak{p}_2 \cap ... \cap \mathfrak{p}_t, g \notin \mathfrak{p}_1$. Then we consider X_g instead. Now we have

$$\mathfrak{p}R_g = \sqrt{(f)}R_g.$$

□