Algebraic surfaces

Blowing-up and Blowing-down

Remark 0.1. *The construction of blowing up can be found almost in any book. (Some called it σ-process however). We refer [Beauville, complex algebraic surfaces, chap. II]. However, Beauville only proved that the map h is a bijective morphism. It would be a good exercise to prove that h indeed an isomorphism.*

In this section, we introduce the important notion of blowing-up. This process is essential in studying singularities and hence birational geometry in general.

We first introduce the local version. Let $\mathbb{A}^n$ be the affine space with coordinates $z_0, ..., z_{n-1}$ and $0 \in \mathbb{A}^n$ be the "origin". We construct a variety $Y \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ by $\{z_i X_j = z_j X_i\}_{i \neq j}$, where $X_0, ..., X_n$ are the homogeneous coordinates of $\mathbb{P}^{n-1}$. There is a natural morphism $\pi : Y \to \mathbb{A}^n$ by projection. One sees that $\pi^{-1}(0) \cong \mathbb{P}^{n-1}$ and $\pi : Y - \pi^{-1}(0) \cong \mathbb{A}^n - \{0\}$. We say Y is the blowing-up of $\mathbb{A}^n$ at 0 and denoted $Bl_0(Y)$.

In general, let $x \in X$ be a point in a variety X. Pick an open affine neighborhood U of x. We identify (U, x) with an open set $(U', 0) \subset \mathbb{A}^n$. Then one has $\widetilde{U} := \pi^{-1}(U') \to U'$ which is the blowing-up of U' at 0. Glue $X - U$ and $\widetilde{U}$ together, we get $\pi_X : \widetilde{X} \to X$. Which is called the blowing-up of X at x. Note that one has similarly that $\pi_X^{-1}(x) \cong \mathbb{P}^{n-1}$ and $\pi_X : \widetilde{X} - \pi_X^{-1}(x) \cong X - \{x\}$. The divisor $\pi_X^{-1}(x)$ is called *the exceptional divisor*, and usually denoted E.

Exercise 0.2. *Let $\pi : X = Bl_x(\mathbb{P}^2) \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at a point $x \in \mathbb{P}^2$. Prove that*

$$K_X = \pi^* K_{\mathbb{P}^2} + E$$

by local coordinate computation.

In fact, if $\dim X = 2$, $\pi : \widetilde{X} \to X$ is a blowing-up at a point $x \in X$, then

$$K_{\tilde{X}} = \pi^* K_X + E.$$

More generally, if $\dim X = n$, and $\pi : \widetilde{X} = Bl_x(X) \to X$ is the blowing-up at x, then

$$K_{\tilde{X}} = \pi^* K_X + (n-1)E.$$

Let's play a little bit around the blowing-ups. Let's restrict ourselves to surfaces. One might expect that there are similar higher-dimensional formulation. Let X be a surface, and $C \subset X$ be a curve. Let f be the local equation of C around x. By fixing local coordinates z_1, z_2, we can write

$$f = f(z_1, z_2) = f_m + f_{m+1} + ...$$

with $f_m \neq 0$. We define the multiplicity of C at x to be

$$m_x(C) := m.$$

One can have an equivalent definition by vanishing order of partial differentials. Hence one can check the $m_x(C)$ is well-defined.

We consider $\pi : \widetilde{X} = Bl_x(X) \to X$. And let C be a curve passing through $x \in X$. Then $\pi^{-1}(C)$ consists of irreducible components, E and the other part maps onto C. The part maps onto C can be defined as

$$\widetilde{C} := \overline{\pi^{-1}(C - \{x\})},$$

which is called the *proper transform of* C. Thus we have $\pi^{-1}(C) = \widetilde{C} \cup E$. More precisely, by computing the equations, one has

$$\pi^* C = \widetilde{C} + m_x(C)E,$$

this is called the *total transform of* C.

We here collect some properties regarding the blowing-up on surface.

Proposition 0.3. *Let* $\pi : \widetilde{X} = Bl_x(X) \to X$ *be the blowing-up at* $x \in X$*. Then one has:*

(1) *There is a natural isomorphism* $Div(X) \oplus \mathbb{Z}E \to Div(\widetilde{X})$ *by* $(D, nE) \mapsto \pi^*D + nE$*. And the isomorphism induces an isomorphism* $\mathrm{Pic}X \oplus \mathbb{Z}E \to \mathrm{Pic}\widetilde{X}$*.*
(2) *Let* $D, D' \in Div(X)$*, then* $(\pi^*D).(\pi^*D') = D.D'$*.*
(3) *Let* $D \in Div(X)$*, then* $(\pi^*D).E = 0$*.*
(4) $E.E = -1$*.*

Proof. It's easy to check the isomorphism given in (1).

For (2) and (3), it follows by choosing Δ, Δ' which are linear equivalent to D, D' respectively but not passing through x.

For (4), by adjunction formula and the fact the $E \cong \mathbb{P}^1$,

$$-2 = deg(K_E) = (K_{\tilde{X}} + E).E = (\pi^*K_X + 2E).E = 2E.E.$$

□

The blowing-up gives the first example of binational morphism.

Definition 0.4. *By a rational map* $f : X \dashrightarrow Y$ *from* X *to* Y*, we mean a regular function on a dense Zariski open (or simply non-empty Zariski open) set* $U \subset X$*.*

More precisely, a rational map can be written as (U, f) *where* $U \subset X$ *is a dense Zariski-open set and* $f : U \to Y$ *is regular.*

We say $(U, f) \sim (V, g)$ *if* $f = g$ *on* $U \cap V$*. In fact, a precise definition of rational map should be the equivalent class of the pairs* (U, f)*. However, we usually abuse the notation if no confusion is likely.*

Definition 0.5. *A rational map* $\phi : X \dashrightarrow Y$ *is said to be birational if it admits an inverse. That is, there is an* $\psi : Y \dashrightarrow X$ *such that* $\psi \circ \phi = id_X, \phi \circ \psi = id_Y$

Example 0.6. *Let* $\pi : Y = Bl_0(\mathbb{A}^2) \to \mathbb{A}^2$*, take* $\psi : \mathbb{A}^2 - \{0\} \to Y \subset \mathbb{A}^2 \times \mathbb{P}^1$ *such that* $\psi(x, y) = ((x, y), [x, y])$*. Then* $\psi \circ \pi = id_Y, \pi \circ \psi = id_X$*. Hence* π *is a birational morphism.*

Exercise 0.7. *The following are equivalent:*

(1) X *and* Y *are birationally equivalent.*
(2) *there are non-empty open subset* $U \subset X$ *and* $V \subset Y$ *such that* U, V *are isomorphic.*
(3) $K(X) \cong K(Y)$ *as* k*-algebra.*

Given a variety X, one can obtain various birational equivalent varieties

$$... \to X_n \to X_{n-1} \to ... \to X_1 \to X$$

by successive blowing-ups. It's also a natural question to ask if X is obtained by blowing-ups? Another way to put it is if X *minimal* or not? The precise formulation of *minimal model in any dimension* is quite subtle.

We start by working on contraction on surfaces. In order to produce a minimal object, we need to tell whether a surface X is obtained from blowing-ups.

Definition 0.8. *Let* $C \subset X$ *be a curve on* X*, we say that* C *is a* (-1)*-curve if* $C \cong \mathbb{P}^1$ *and* $C^2 = -1$

We seen that we can have a (-1)-curve by blowing-up. In fact we will prove that any (-1)-curve comes from blowing-ups.

Theorem 0.9 (Caltelnuovo)**.** *Let X be a surface (non-singular complex projective surface) with $E \subset X$ a (-1)-curve. Then there is a morphism $\pi : X \to X'$ with X' non-singular such that π is the blowing-up of X' with exceptional divisor E.*

Proof. The idea is to construct a morphism which is identical at E but isomorphic outside E.

First pick H' any very ample divisor on X, Let $k := H.E > 0$. We consider $H' = H + kE$, then $H'.E = 0$. Notice that the restriction map

$$H^0(X, \mathcal{O}(H')) \to H^0(E, \mathcal{O}(H'|_E) = \mathcal{O}_E) \cong H^0(\mathbb{P}^1, \mathcal{O}) \cong \mathbb{C}.$$

Hence the map $\varphi_{H'}$ produce by $|H'|$ is constant on E. We need to refine H so that $\varphi_{H'}$ is isomorphic outside E.

To this end, we first pick any very ample H_0. It's clear that nH_0 is very ample for all $n > 0$. On the other hand, H_0 is ample, one can arrange that $H := nH_0$ is very ample with $H^1(X, \mathcal{O}(H)) = 0$.

Consider the exact sequence

$$0 \to \mathcal{O}_X(H + (i-1)E) \to \mathcal{O}_X(H + iE) \to \mathcal{O}_E(H + iE|_E) = \mathcal{O}_E(k - i) \to 0.$$

Claim. $H^1(X, \mathcal{O}(H + iE)) = 0$ for all $1 \leq i \leq k$.

Grant this for the time being, then one has an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(H + (i-1)E)) \to H^0(X, \mathcal{O}_X(H + iE)) \to H^0(E, \mathcal{O}_E(k - i)) \to 0.$$

Note that $H^0(E, \mathcal{O}_E(k-i))$ is of dimension $k - i + 1$, let $a_{i,0}, ..., a_{i,k-i} \in H^0(X, \mathcal{O}(H + iE))$ be the lifting of a basis in $H^0(E, \mathcal{O}(k-i))$.

Remark. Before we move on, we would like to remark the difference between $H^0(X, \mathcal{O}(D))$ and $\mathcal{L}(D)$. It actually comes from two possible definition of $\mathcal{O}(D)$. If we define the sheaf $\mathcal{O}(D)$ as $\mathcal{O}(D)(U) = \{f \in K(X) | div(f) + D|_U \geq 0 \text{ on } U\}$. Then $H^0(X, \mathcal{O}(D)) = \mathcal{L}(D)$. However, another way to look at the sheaf $\mathcal{O}(D)$ is to consider it as the sheaf of sections line bundle associate to D. Then under this consideration, for $s \in H^0(X, \mathcal{O}(D))$, $div(s)$ gives an effective divisor D_s linearly equivalent to D. To view it as $\mathcal{L}(D)$ is the classical treatment. The Modern viewpoint tends to think it as section of line bundles. We take the convention that $H^0(X, \mathcal{O}(D))$ represents the global section of line bundle of D from now on.

Let me describe the correspondence in more detail. Given a divisor D, one has a system of local equations $(U_i f_i)$. The basic idea behind the notion of line bundle is instead of looking at functions, we look at local functions satisfying given patching conditions. The correspondence is given as

$$\mathcal{L}(D) \to H^0(X, \mathcal{O}(D)),$$

$$f \mapsto (U_i, ff_i) = s.$$

And the correspondence between their divisor is given by

$$div(s) = div(f) + D,$$

which is an effective divisor $D_s \in |D|$.

Turning back to the proof, let $s \in H^0(X, \mathcal{O}(E))$ be a section such that $div(s) = E$. Then the map $H^0(X, \mathcal{O}_X(H + (i-1)E)) \to H^0(X, \mathcal{O}_X(H + iE))$ is given by multiplying s. Therefore, by working on the sequence inductively, one can have a basis of $H^0(X, \mathcal{O}(H + kE))$, given as

$$\{s_0 s^k, ..., s_n s^k, a_{1,0} s^{k-1}, ..., a_{1,k-1} s^{k-1}, ..., a_{k-1,0} s, a_{k-1,1} s, a_k\}.$$

We consider the map $\varphi_{H'} : X \to \mathbb{P}^N$ given by the above basis. Note that $a_k \in H^0(X, \mathcal{O}(H'))$ whose restriction to E is a non-zero constant. Hence one has

φ is well-defined along E and $\varphi(E) = [0, ..., a_k] = [0, ..., 1]$. Moreover, for $x \notin E$, $s(x) \neq 0$, hence

$$[s_0 s^k(x), ..., s_n s^k(x)] = [s_0(x), ..., s_n(x)] = \varphi_H(x).$$

Since H is very ample, φ_H defines an embedding on X and hence on $X - E$. One sees that the first $n+1$ coordinate of $\varphi_{H'}$ gives an embedding on $X - E$ already, so it follows that $\varphi_{H'}$ gives an embedding on $X - E$.

It remains to show that $X' := \varphi_{H'}(X)$ is non-singular. Let $U \subset X$ be the open subset defined by $a_k \neq 0$. It's clear that $E \subset U$. We want to identify U with an open set $V \subset \widetilde{\mathbb{A}^2} \subset \mathbb{A}^2 \times \mathbb{P}^1$. This can be achieved by considering

$$h : U \to \mathbb{A}^2 \times \mathbb{P}^1,$$
$$x \mapsto \big((\frac{a_{k-1,0} s}{a_k}(x), \frac{a_{k-1,1} s}{a_k}(x)), [a_{k-1,0}(x), a_{k-1,1}(x)] \big).$$

We might need to shrink U so that $a_{k-1,0}(x)$ and $a_{k-1,1}(x)$ are not simultaneously vanishing. It's obvious that h factor through $\widetilde{\mathbb{A}^2}$. Let $V = h(U) \subset \widetilde{\mathbb{A}^2}$. Moreover, one has the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & \widetilde{\mathbb{A}^2} \\ {\scriptstyle \varphi_{H'}}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ \varphi_{H'}(U) & \xrightarrow{\bar{h}} & \mathbb{A}^2, \end{array}$$

where $\bar{h} = (\frac{a_{k-1,0} s}{a_k}, \frac{a_{k-1,1} s}{a_k})$ is a rational map on $\mathbb{P}^N$ defined on $\varphi_{H'}(U)$. Another remark is that h clearly map $E \subset U$ onto $E \subset \widetilde{\mathbb{A}^2}$. It suffices to show that $h : U \to V$ is an isomorphism. Because, the induced map $\bar{h}$ is an isomorphism . Therefore, $\varphi_{H'}(U)$ is non-singular at $\varphi_{H'}(E)$, which is the only possible singularity.

However, to show that h is an isomorphism is not trivial. One can first prove that it's a hemeomorphism, hence in particular, bijective. Then one prove the h induces isomorphism on all local rings. (cf. [Ha. Ex I.3.2, I.3.3]) $\square$