Algebraic surfaces

Bertini's theorem, Ampleness Criterion, Intersection theory and Riemann-Roch theorem on surfaces

Remark 0.1. *Please refer to [Ha, II 7] for ampleness and very ampleness. Another source is Hartshorne's book: Ample subvarieties of algebraic varieties, Lecture Notes in Mathematics 156. The discussion of intersection can be found in [Ha, V 1], while Riemann-Roch appears in [Ha, appendix A]. The proof of Bertini's theorem is the one in [G-H, p137]=[Griffiths, Harris]. For more discussion on cone of curves, we refer [Kollár, Mori]: Birational geometry of algebraic varieties. In which they give a complete treatment of minimal model program.*

Let D be a divisor on X. If $D' \sim D$, then $\mathcal{O}(D) \cong \mathcal{O}(D')$. Thus sometime it's useful to pick a better element in $|D|$ instead of looking at D itself.

Theorem 0.2 (Bertini)**.** *Let $Bs|D|$ be the base locus of $|D|$. If $\dim|D| \geq 1$, then the general member of $|D|$ is non-singular away from the $Bs|D|$.*

In particular, if $|D|$ is base point free, then general member in $|D|$ is non-singular.

Proof. Fix $D_0, D_1 \in |D|$ with local equation $f, f+g$ on an affine open set $U \subset X$. We have a subseries (a pencil) $D_\lambda \in |D|$ locally defined by $f + \lambda g$. Suppose that $P_\lambda \in U$ is a singular point of D_λ not in the base locus $B := Bs|D|$. We may assume that $g(P_\lambda) \neq 0$. We have

$$f + \lambda g(P_\lambda) = 0,$$

and

$$\frac{\partial}{\partial z_i}(f + \lambda g)(P_\lambda) = 0,$$

for all z_i. Where $z_1, ..., z_n$ are the local coordinates.

These equations defines a subvariety in $Z \subset U \times \mathbb{P}^1$. And let $V = pr_1(Z) \subset U$.

One note that f/g is locally constant $(-\lambda)$ on $V - B$. Hence for λ different from value of f/g, D_λ is non-singular away from B (in U).

Next one notice that one can cover X by finitely many affine open sets. □

Remark 0.3. *If $|D|$ is very ample, then it is base point free.*

Definition 0.4. *A divisor D is said to be ample if mD is very ample for some* $m > 0$.

Our first aim is the following:

Theorem 0.5. *Let D be a divisor on a projective variety X. There exist a very ample divisor A such that $A + D$ is very ample.*

Corollary 0.6. *Let D be a divisor on a projective variety X. Then there are non-singular very ample divisor Y_1, Y_2 such that $D \sim Y_1 - Y_2$*

Lemma 0.7. *The following are equivalent:*

(1) *D is ample.*
(2) *For every coherent sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F} \otimes \mathcal{O}(nD)) = 0$ for all $i > 0$ and $n \gg 0$.*
(3) *For every coherent sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{O}(nD)$ is globally generated for all $n \gg 0$.*

Remark 0.8. *A sheaf is globally generated if the natural map $H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$ is surjective. If $\mathcal{F} = \mathcal{O}(D)$ for some divisor D, then $\mathcal{O}(D)$ is globally generated if and only if D is base point free.*

Exercise 0.9. *Show that if D_1 is very ample and D_2 is base point free, then D_1+D_2 is very ample.*

(Hint: consider the subspace $\mathcal{L}(D_1)\otimes\mathcal{L}(D_2) \subset \mathcal{L}(D_1+D_2)$. Show that the map defined by this subspace is everywhere defined and an embedding. Thus the map defined by D_1+D_2 is an embedding.

proof of theorem 0.5. X is projective, then $X \hookrightarrow \mathbb{P}^n$ for some n. Take H a hyperplane in $\mathbb{P}^n$, then $H \cap X$ is a very ample divisor on X. By abuse the notation, we still called it,denoted H ,a hyperplane section.

Note that very ample is clearly ample. Hence by the Lemma 0.8 (3), there is an n_0 such that $D+n_0H$ is base point free. By the exercise, $D+(n_0+1)H$ is very ample. □

We are now able to define intersection of subvarieties. We start by considering intersection on surface.

Theorem 0.10. *Let X be a non-singular projective surface. There is a unique pairing $Div(X) \times Div(X) \to \mathbb{Z}$, denoted by $C.D$ for any two divisor C, D, such that*

(1) *if C and D are non-singular curves meeting transversally, then $C.D = \#(C \cap D)$,*
(2) *it is symmetric. i.e. $C.D = D.C$,*
(3) *it is additive. i.e. $(C_1+C_2).D = C_1.D + C_2.D$,*
(4) *it depends only on the linear equivalence classes. i.e. if $C_1 \sim C_2$ then $C_1.D = C_2.D$.*

Proof. See [Ha, V 1.1]. □

Remark 0.11. *Let X be a projective variety. An 1-cycle is a formal linear combination of irreducible curves. The group of all 1-cycles is denoted $Z_1(X)$ (=free abelian group on irreducible curves). One can similarly define a pairing $Z_1(X) \times Div(X) \to \mathbb{Z}$.*

Two curves C_1, C_2 are said to be numerically equivalent if $C_1.D = C_2.D$ for all D, denoted $C_1 \equiv C_2$. We define

$$N_1(X) := Z_1(X)\otimes\mathbb{R}/\equiv .$$

It's a famous theorem asserts that $N_1(X)$ is finite dimensional. Its dimensional is called Picard number, denoted $\rho(X)$.

Remark 0.12. *Let $V \subset X$ be a subvariety of codimension i, and D is a divisor. Then it make sense to consider $V.D^i$ by decomposing $D \sim H_1 - H_2$ and then compute $(V \cap H_i).D^{i-1}$ in $V \cap H_i$ inductively on dimension. One can simply set*

$$V.D^i := (V \cap H_1).D^{i-1} - (V \cap H_2).D^{i-1}.$$

Remark 0.13. *Let X be a variety over $\mathbb{C}$. A divisor D gives a class $c_1(D) \in H^2(X, \mathbb{Z})$ via $Div(X) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$. And a curve C give rise to a class $[C] \in H_2(X, \mathbb{Z})$. The pairing $H_2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$ gives an intersection theory.*

An important feature of ampleness is that it's indeed a "numerical property".

Theorem 0.14 (Nakai's criterion)**.** *Let X be a projective variety. A divisor D is ample if and only $V.D^i > 0$ for all subvariety of codimension i.*

In particular, if $\dim X = 2$, then D is ample if and only if $D.D > 0$ and $D.C > 0$ for all irreducible curve C.

Another important criterion is due to Kleiman. Let $NE(X) \subset N_1(X)$ be the cone generated by effective curves. And let $\overline{NE(X)}$ be its closure.

For any divisor D, it defines a linear functional on $N_1(X)$ and we set $D_{>0} = \{x \in N_1(X) | (x.D) > 0\}$.

Theorem 0.15 (Kleiman's criterion). *D is ample if and only if*

$$D_{>0} \supset \overline{NE(X)} - \{0\}.$$

Before we revisit the Riemann-Roch theorem on surface, we need the useful adjunction formula:

Proposition 0.16 (Adjunction formula). *Let $S \subset X$ be a non-singular subvariety of codimension 1 in a non-singular variety X. Then $K_S := K_X + S|_S$. In particular, if $\dim X = 2$ then $2g(S) - 2 = (K_X + S).S$*

Given a codimension 1 subvariety $Y \subset X$ and a divisor $D \in Div(X)$. One can consider the restriction $D|_Y$, which is supposedly to be a divisor. However, this is not totally trivial. For $D = \sum n_i D_i$, one might want to consider naively that $D|_Y := \sum n_i(D_i \cap Y)$. But what if $D_i = Y$ for some i? That is, how to define $Y|_Y$?

One way to think of this is that we deform Y such that $\lim_{t\to 0} Y_t = Y$, then we take $Y|_Y := \lim_{t\to 0} Y_t|_Y$. (This needs some extra care).

proof of adjunction formula. Recall that a canonical divisor is a divisor defined by n-forms if $\dim X = n$. Thus one has $\Omega^n_X \cong \mathcal{O}_X(K_X)$. Where Ω^n_X denote the sheaf of n-forms on X. Also one can consider sheaf of n-forms on X with pole along S, denoted $\Omega^n_X(S)$. It's clear that $\Omega^n_X(S) \cong \mathcal{O}_X(K_X + S)$.

One has the following exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + S) \to \mathcal{O}_S(K_X + S|_S) \to 0.$$

On the other hand, one has the Poincaré residue map

$$\Omega^n_X(S) \twoheadrightarrow \Omega^{n-1}_S$$

with kernel Ω^n_X. Comparing these two sequences, one sees that $\Omega^{n-1}_S \cong \mathcal{O}_S(K_X + S|_S)$. Hence the canonical divisor $K_S = K_X + S|_S$.

We now describe the Poincaré residue map. (cf. [G-H, p147]). The problem is local in nature, it suffices to describe it locally. We may assume that on a small open set U, S is defined by f. And let $z_1, .., z_n$ be the local coordinates of U.

The sheaf $\Omega^n_X(S)$ on U can be written as $\omega = \frac{g(z)dz_1\wedge ...\wedge d_n}{f(z)}$. Since S is non-singular, then at least one of $\frac{\partial f}{\partial z_i} \neq 0$. The residue map send ω to

$$\omega' := (-1)^{i-1}\frac{g(z)dz_1 \wedge ... \wedge \widehat{dz_i} \wedge ... \wedge d_n}{\partial f/\partial z_i} |_{f=0} .$$

This is independent of choice of i since on S

$$df = \frac{\partial f}{\partial z_1}dz_1 + ... + \frac{\partial f}{\partial z_n}dz_n = 0.$$

Another way to put it is that the residue map sends ω to ω' such that $\omega = \frac{df}{f} \wedge \omega'$.

It's clear that the $\omega' = 0$ if and only if $f(z)|g(z)$, which means that ω is indeed in Ω^n_X.

$\square$

Theorem 0.17 (Riemann-Roch theorem for divisors on surfaces). *Let X be a non-singular projective surface and $D \in Div(X)$ a divisor on X, then one has*

$$\chi(X, D) = \chi(X, \mathcal{O}_X) + \frac{1}{2}D.(D - K_X).$$

Proof. Write $D \sim H_1 - H_2$ with H_i are non-singular very ample divisor. We consider the sequences:

$$0 \to \mathcal{O}(D) \cong \mathcal{O}(H_1 - H_2) \to \mathcal{O}(H_1) \to \mathcal{O}_{H_2}(H_1) \to 0,$$
$$0 \to \mathcal{O} \to \mathcal{O}(H_1) \to \mathcal{O}_{H_1}(H_1) \to 0.$$

It's clear that

$$\chi(X, D) = \chi(X, H_1) - \chi(H_2, \mathcal{O}_{H_2}(H_1))$$
$$= \chi(X, \mathcal{O}_X) + \chi(H_1, \mathcal{O}_{H_1}(H_1)) - \chi(H_2, \mathcal{O}_{H_2}(H_1)).$$

By Riemann-Roch on curves and adjunction formula,

$$\chi(H_1, \mathcal{O}_{H_1}(H_1)) = H_1.H_1 + 1 - g(H_1) = H_1.H_1 + 1 - \frac{1}{2}(K_X + H_1).H_1,$$

$$\chi(H_2, \mathcal{O}_{H_2}(H_1)) = H_1.H_2 + 1 - \frac{1}{2}(K_X + H_2).H_2.$$

Collecting terms, one has

$$\chi(X, D) = \chi(X, \mathcal{O}_X) + \frac{1}{2}(H_1 - H_2).(H_1 - H_2 - K_X) = \chi(X, \mathcal{O}_X) + \frac{1}{2}D.(D - K_X).$$

□