# Algebraic surfaces

## LINEAR SERIES AND MAPS TO PROJECTIVE SPACES

Fix a non-singular variety $X$ and a divisor $D \in Div(X)$. One can define

$$\mathcal{L}(D) := \{f \in K(X) | div(f) + D \geq 0\},$$

and

$$|D| = \{D' \in Div(X) | D' \sim D, D' \geq 0\}.$$

Moreover there is a map $\pi : \mathcal{L}(D) - \{0\} \to |D|$ by sending $f$ to $div(f) + D$. Such $|D|$ is called the complete linear series of $D$. One can also consider a vector subspace $V \subset \mathcal{L}(D)$ and then produce a subseries $|V| \subset |D|$.

Another remark is that $\pi$ maps the punctured space to a projective space. Thus $\dim|D| = \dim\mathcal{L}(D) - 1$.

Suppose now that $\mathcal{L}(D) \neq \{0\}$, one can define a map $X \dashrightarrow \mathbb{P}^n$. To start with, we pick a basis $f_0, f_1, ..., f_n$ of $\mathcal{L}(D)$ and consider the local equation of $D$ , denoted $\{(f_\alpha, U_\alpha)\}$. For $x \in U_\alpha$, we define $\varphi_\alpha : x \mapsto [f_0 f_\alpha(x), f_1 f_\alpha(x), ..., f_0 n f_\alpha(x)]$.

One notice that on $U_\alpha$, $div(f_i) + D|_{U_\alpha} = div(f_i f_\alpha) \geq 0$. Hence $f_i f_\alpha$ is indeed a regular function on $U_\alpha$. It turns out the $\varphi_\alpha$ is undefined only at the common zero of $f_i f_\alpha$.

If $x \in U_\alpha \cap U_\beta$, then $f_\alpha f_\beta^{-1}(x)$ is a non-zero constant, hence $\varphi_\alpha(x) = \varphi_\beta(x)$. All these maps patch together to give $\varphi_D : X \dashrightarrow \mathbb{P}^n$. Let $Bs|D|$ the the locus where $\varphi_D$ is undefined. Then $Bs|D|$ can be described as

$$\begin{aligned} Bs|D| &= \{x \in X | f_i f_\alpha(x) = 0 \quad \forall i, \text{ for some } U_\alpha \ni x\} \\ &= \{x \in X | f f_\alpha(x) = 0, \quad \forall f \in \mathcal{L}, \text{ for some } U_\alpha \ni x\} \\ &= \{x \in X | x \in Supp(D'), \quad \forall D' \in |D|\}. \end{aligned}$$

The set $Bs|D|$ is call the base locus of $|D|$.

**Definition 0.1.** *$|D|$ (or $D$) is said to be base point free if $Bs|D| = \emptyset$. In this case, $\varphi_D$ is a morphism.*
*$|D|$ (or $D$) is said to be very ample if $\varphi_D$ is a embedding.*

**Example 0.2.** *Consider $X = \mathbb{P}^2$ and $D = (XY - Z^2 = 0) \in Div(X)$. Then $\mathcal{L}(D)$ has a basis $f_0 := \frac{XY}{XY-Z^2}, f_1 := \frac{YZ}{XY-Z^2}, f_2 := \frac{ZX}{XY-Z^2}, f_3 := \frac{X^2}{XY-Z^2}, f_4 := \frac{Y^2}{XY-Z^2}, f_5 := \frac{Z^2}{XY-Z^2}$*

*If we consider the subspace $V =< f_0, f_1, f_2 >$, then $\varphi_{|V|} : X \dashrightarrow \mathbb{P}^2$ by $[a_0, a_1, a_2] \mapsto [a_0 a_1, a_1 a_2, a_2 a_0]$. One finds that $Bs|V| = \{[1,0,0], [0,1,0], [0,0,1]\}$.*

*If we consider the complete linear series $|D|$, then $\varphi_D : X \to \mathbb{P}^5$ by $[a_0, a_1, a_2] \mapsto [a_0 a_1, a_1 a_2, a_2 a_0, a_0^2, a_1^2, a_2^2]$ without any base point. Hence $|D|$ is base point free. In fact, it's very ample.*