# Algebraic surfaces

## Minimal model program on surfaces

The previous result shows that a minimal model for surface always exists and that a surface $X$ is minimal if and only if $X$ has no $(-1)$-curve. Hence we may and we usually do assume that $X$ is minimal. However, the criterion by $(-1)$ curve only valid for surface. We might prefer to have a criterion which is also valid for higher dimension.

If $X$ has a $(-1)$-curve $E$, then $K_X.E = -1 < 0$. Thus if $K_X$ is nef then $X$ has no $(-1)$-curve. Hence $X$ is minimal. The minimal model program (or sometimes called *Mori's program* can be describe as a program to find minimal models. The important criterion is nefness of $K_X$. Let's start with a variety $X$, if $K_X$ is nef, then we have a minimal model and stop here. If $K_X$ is not nef, then there is a curve $C$ such that $C.K_X < 0$. Then one can produce a morphism $f : X \to Y$ contracting $C$ (and possibly contracting more curves at the same time). If $\dim Y < \dim X$, then the morphism has some special structure which is called *Mori's fibration*. And the program stop here.

If $\dim X = \dim Y$ and $f$ contracts a divisor (we called it *divisorial contraction*). Then $\rho(Y) < \rho(X) := rk(NS(X))$. We replace $X$ by $Y$ and start the program over again. By looking at $\rho(X)$, it can't be an infinite loop, that is, the program must stop.

The remaining is the subtle one. If $\dim X = \dim Y$ and $f$ contracts a subvariety of codimension $\geq 2$ . Then call it a *small contraction*. One needs *flips* to produce another birational model $f' : X' \to Y$. However, there is no infinite sequence of flips. Thus one must stop at somewhere $f : \tilde{X} \to \tilde{Y}$ which doesn't allow flips. Thus replacing $X$ by $\tilde{X}$ then it must go to other cases.

For surface, we don't need to worry about small contraction. Therefore, by running the minimal model program, the resulting products are surfaces with $K_X$ nef and Mori fibration over a curve or a point.

**Theorem 0.1.** *Let $X$ be a minimal surface, then either $K_X$ is nef or $X$ is a ruled surface or $\mathbb{P}^2$. In fact, $X$ is $\mathbb{P}^2$ when $\rho(X) = 1$ and $X$ is ruled when $\rho(X) \geq 2$.*

We need the following two highly non-trivial facts:

(1) If $K_X$ is not nef, then there exists a rational curve $C \cong \mathbb{P}^1$ such that $C.K_X < 0$.
(2) Fix an ample divisor $H$, there is a rational curve $C$ with $K_X.C < 0$ such that $\frac{-K_X.C}{H.C}$ is maximal.

The point for the first fact is that if $K_X.C < 0$ then by *reduction to characteristic $p$*, one sees that the curve $C$ can be deformed (in char $p$). Thus one has a morphism $F : C \times \mathbb{A}^1 \to X$. The morphism extends to a rational map $\bar{F} : C \times \mathbb{P}^1 \dashrightarrow$. By *rigidity lemma*, one shows that

$\bar{F}$ can't be a morphism, i.e. must have point of indeterminacy. We then eliminate the indeterminacy by blowing-ups to get a morphism $\tilde{F} : Y \to X$. The exceptional curve $E \cong \mathbb{P}^1$ then maps to a rational curve in $X$, we denote it by $E$. Moreover, $C \equiv C' + E$, thus either $E.K_X < 0$ or $C'.K_X < 0$. If $K_X.E < 0$ then we are done, otherwise, we replace $C$ by $C'$. With arithmetic genus $p_a(C') < p_a(C)$, we must stop somewhere and get a rational curve.

The idea for proving the second fact is more subtle, it's basically the *rationality theorem.*

Before we get into the proof, we would like to define the *arithmetic genus* which will be useful in the sequel.

**Definition 0.2.** *Let $D$ be an effective divisor in a surface $X$, then we define the arithmetic genus*

$$p_a(D) := \frac{1}{2}(D^2 + K_X.D) + 1.$$

*Note that if $D$ is a non-singular curve, then $p_a(D) = g(D)$.*

Let $C \subset X$ be a possibly singular curve. By blowing-up on X along singularities of $C$, one has proper transform $\widetilde{C} \subset \widetilde{X} \to X$ which is non-singular. We leave it as an exercise to show that $p_a(\widetilde{C}) \leq p_a(C)$ and $<$ holds if $C$ is singular. Nevertheless, $p_a(\widetilde{C}) = g(\widetilde{C}) \geq 0$. Therefore we have:

**Proposition 0.3.** *Let $C \subset X$ be a possibly singular curve. Then $p_a(C) \geq 0$.*

*And if $p_a(C) = 0$, then $C$ is non-singular and $C \cong \mathbb{P}^1$.*

*proof of the theorem.* Assume those facts, we have a a rational curve $C$ with $K_X.C < 0$ such that $\frac{-K_X.C}{H.C}$ is maximal, where $H$ is a fixed very ample divisor. Let $\frac{q}{p}$ be the maximal value. Let $D := pK_X + qH$, it's clear that $D.C = 0$ and $D.C' \geq 0$ for any irreducible curve $C'$. In particular, $D$ is nef.

**Remark.** If $D$ is a nef divisor on a surface, then $D.C \geq 0$ for all curves and $D^2 \geq 0$.

We first take care of the case that $\rho(X) \geq 2$.

**Claim 1.** $h^0(X, \mathcal{O}_X(mD)) > 0$ for $m \gg 0$.

**Claim 2.** $|mD|$ is base point free for $m \gg 0$.

Grant these for the time being, we then fix an $m_0 \gg 0$ such that $|m_0 D|$ is free. We have a morphism

$$\varphi_{m_0 D} : X \to \varphi(X) =: Y \subset \mathbb{P}^n.$$

Note that the restriction

$$H^0(X, \mathcal{O}(m_0 D)) \to H^0(C, \mathcal{O}(m_0 D|_C) = \mathcal{O}) \cong \mathbb{C},$$

gives constant functions. One concludes that the morphism $\varphi$ maps $C$ to a point.

**Facts we need.** Another fact we need is that $Y$ is non-singular for $m \gg 0$. Then by minimality of $X$, $\dim Y < \dim X$. If $\rho(X) \geq 2$, then one can conclude that there is a curve $C'$ with $D.C' > 0$.

**Claim 3.** We may assume that the restriction

$$H^0(X, \mathcal{O}(mD)) \to H^0(C', \mathcal{O}(mD|_{C'}))$$

is non-constant.

As a result, the restriction of $\varphi$ to $C'$ is not constant and so $\varphi$ is not constant. Hence $\dim Y \geq 1$. So $\dim Y = 1$. We may assume that $\varphi : X \to Y$ is a fibration, i.e. surjective with connected fibers. Moreover, a general fiber is a non-singular curve.

It remains to analyze the structure of $\varphi$. Especially, we wish to prove that the fiber is $\cong \mathbb{P}^1$. We need the famous

**Zariski Lemma.** Let $\pi : X \to B$ be a fibration from a surface to a curve. Let $F_s = \sum_i n_i C_i$ be a fiber and $D = \sum_i m_i D_i$ with $m_i \geq 0$ for all $i$. Then $D^2 \leq 0$. In particular, $C_i^2 \leq 0$ for all $i$.

Let's look at the fibration $\varphi : X \to Y$. We hope to prove that every fiber $F_s \cong \mathbb{P}^1$. Let $C_0 \subset F_s$ be an irreducible component. As we have seen, the curve $C_0$ contains in a fiber if and only if $D.C_0 = 0$. Hence $K_X.C_0 < 0$. Moreover, by Zariski Lemma, $C_0^2 \leq 0$. By adjunction formula,

$$-2 \leq 2p_a(C_0) - 2 = K_X.C_0 + C_0.C_0 < 0.$$

The only possibility is $C_0^2 = 0, K_X.C_0 = -2$ since $X$ has no $(-1)$-curve.

Let $F_s := \varphi^*(s) = \sum_i n_i C_i$ be a fiber of $\varphi$. It's clear that $F_s^2 = 0$. And we have seen that $C_i^2 = 0$. It follows that

$$0 = F_s^2 = 2\sum_{i \neq j} n_i n_j C_i C_j \geq 0.$$

Since $F_s$ is connected, if there are more than two components in $F_s$, then $C_i.C_j > 0$ for some $i \neq j$ which is a contradiction. Therefore $F_s$ is irreducible, i.e. say $F_s = n_s C_s$.

For $s \neq t \in B$,

$$-2n_s = F_s.K_X = F_t.K_X = -2n_t.$$

It turns out that $n_s = n_t$ for all $s, t \in B$. However, for general fiber $F$ is a non-singular curve. One has $n_s = 1$ for all $s$. This completes the proof of the case that $\rho(X) \geq 2$.

*proof of the claims.* In order to prove the claims, we need

**Kodaira Vanishing Theorem.** Let $X$ be a non-singular projective variety over $\mathbb{C}$. Let $L$ be an ample divisor, then

$$H^i(X, \mathcal{O}_X(K_X + L)) = 0 \quad \forall i > 0.$$

To prove the Claim 1, we consider

$$mD - K_X = (mp-1)K_X + mqH \equiv \frac{mp-1}{p}D + (mq - \frac{(mp-1)q}{p})H.$$

Since $D$ is nef and $H$ is ample. It's clear that "nef+ample is ample". Hence $mD-K_X$ is ample for all $m > 0$. By Kodaira vanishing theorem, one has

$$\chi(X, \mathcal{O}(mD)) = h^0(X, \mathcal{O}(mD)).$$

By Riemann-Roch,

$$h^0(X, \mathcal{O}(mD)) = \chi(X, \mathcal{O}_X) + \frac{1}{2}(mD - K_X).mD.$$

It suffices to prove that $D.H > 0$ for any ample divisor since $mD - K_X$ is ample.

Suppose on the contrary that $D.H = 0$, (recall that $D.C' > 0$ for some $C'$, so $D \not\equiv 0$.) By Hodge Index Theorem, $D^2 < 0$. This contradicts to $D$ being nef. ($D$ is nef implies that $D^2 \geq 0$.) This completes the proof of Claim 1.

To prove the Claim 2. We remark that the following conditions are equivalent.

(1) $x$ is a base point of $|D|$.
(2) Every section of $H^0(X, \mathcal{O}(D))$ vanishing at $x$.
(3) The evaluation map $H^0(X, \mathcal{O}(D)) \to \mathbb{C}(p)$ is zero.
(4) The natural mao $H^0(X, \mathcal{O}(D) \otimes \mathcal{I}_x) \to H^0(X, \mathcal{O}(D))$ is an isomorphism.
(5) $H^1(X, \mathcal{O}(D) \otimes \mathcal{I}_x) \neq 0$

Where $\mathcal{I}_x$ denotes the ideal sheaf of $x$ and $\mathcal{O}(D) \otimes \mathcal{I}_x$ is obtained by considering sections in $\mathcal{O}(D)$ vanishing along $x$. Therefore, in order to prove the base point freeness, it's enough to prove that $H^1(X, \mathcal{O}(mD) \otimes \mathcal{I}_x) = 0$. One might want to apply Kodaira vanishing theorem to prove $H^1 = 0$, however, it only works for divisor. Therefore, we consider $\pi : X' = Bl_x(X) \to X$. It's not too difficult (but not trivial) to see that

$$H^1(X', \mathcal{O}(\pi^* mD - E)) \cong H^1(X, \mathcal{O}(mD) \otimes \mathcal{I}_x).$$

Consider now $L_m := \pi^* mD - E - K_{X'} = \pi^*(mD - K_X) - 2E$. We leave it as an exercise to show that $L_m$ is ample for $m \gg 0$. Then by Kodaira vanishing theorem, we are done.

To prove the last claim, it suffices to prove that $|mD|$ separate two general points on $C'$. To this end, we first fixed $x \in C' \subset X$. We consider the linear series $|mD \otimes \mathcal{I}_x|$ which is a subseries of $|mD|$ consisting of those divisors passing through $x$. As long as $\dim |mD \otimes \mathcal{I}_x| \geq 1$ then $Bs|mD \otimes \mathcal{I}_x|$ is finite. We pick any $y \notin Bs|mD \otimes \mathcal{I}_x|$. Therefore, a general member $D' \in |mD \otimes \mathcal{I}_x|$ passing through $x$ but not $y$. Hence the corresponding section $s \in H^0(X, \mathcal{O}(mD))$ has the property that

$s(x) = 0, s(y) \neq 0$. In particular, we have proved that Claim 3. It follows that $\varphi(x) \neq \varphi(y)$. $\square$

The remaining case is to show that a minimal surface with $\rho(X) = 1$ is $\mathbb{P}^2$. This might require some characterization of $\mathbb{P}^2$ which we will prove later. $\square$