Advanced Algebra I

Midterm

Nov. 22, 2003

(1) (15 pts) Let G be a group of order 500. Show that G is solvable. And give two examples of non-abelian groups of order 500 which are not isomorphic. (You need to verify that your groups are non-isomorphic.)

(2) (15 pts) Let G be a solvable group and $N \lhd G$ be a normal subgroup. Show that G/N is solvable.

(3) (10 pts) Let $\mathrm{Aut}(G)$ be the group of automorphisms of G. Prove that if $\mathrm{Aut}(G)$ is cyclic, then G is abelian.

(4) (10 pts) Let $\mathbb{F}_q$ be a finite field of q elements. Determine the number of elements in $\mathrm{GL(n, \mathbb{F}_q)}$,$\mathrm{PGL(n, \mathbb{F}_q)}$,$\mathrm{SL(n, \mathbb{F}_q)}$,$\mathrm{PSL(n, \mathbb{F}_q)}$.
(Remark: if you don't know how to do it in general, you can work on the case that $n = 2$ to get partial credits)

(5) (15 pts) Let G be a group of order p^3, where p is a prime, and G is non-abelian.
   (a) Show that the center $Z(G) \cong \mathbb{Z}_p$ and $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
   (b) Let H be a subgroup of order p^2. Show that $Z(G) \subset H$ and H is normal.

(6) (15 pts) Consider the group D_4.
   (a) Determine the character table of D_4.
   (b) Find the unique irreducible representation of degree 2 (up to isomorphism).
   (c) Let ρ be an irreducible representation of degree 2. Decompose $\rho \otimes \rho$ into irreducible representations up to isomorphism.

(7) (20 pts) suppose that G is a finite group with the following partial character table:

	(1) 1	(4) a	(5) b	(5) c	(5) d
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	$-i$	i	-1
χ_4	1	1	i	$-i$	-1

   (a) Determine the missing row(s) of the character table.
   (b) Show that G has a normal subgroup of order 10. And determine the structure of this subgroup.
   (c) Determine the structure of Sylow 2-subgroups.
   (d) Determine the structure of G.