# Advanced Algebra I

## REPRESENTATION OF FINITE GROUPS

Another interesting realization of the tetrahedral group $T$ is done by choose coordinates such that $\pm e_i$ are those midpoint of 6 edges. Then one can express $T$ as a finite subgroup of $GL(3, \mathbb{R})$. This is an example of a representation.

**Definition 0.1.** *A $n$-dimensional matrix representation of a group $G$ is a homomorphism*

$$R : G \to GL(n, F),$$

*where $F$ is a field. A representation is faithful if $R$ is injective. And we write $R_g$ for $R(g)$*

It's essential to work without fixing a basis. Thus we introduce the concept of representation of a group on a finite dimensional vector space $V$.

**Definition 0.2.** *By a representation of $G$ on $V$, we mean a homomorphism $\rho : G \to GL(V)$, where $GL(V)$ denote the group of invertible linear transformations on $V$. We write $\rho_g$ for $\rho(g)$*

**Remark 0.3.** *By fixing a basis $\beta$ of $V$, one has*

$$\beta : GL(V) \to GL(n, F)$$

$$T \mapsto \text{matrix of } T.$$

*And one has a matrix representation $R := \beta \circ \rho$.*

*Furthermore, if a change of basis is given by a matric $P$, then one has the conjugate representation $R' = PRP^{-1}$, that is $R'_g = PR_gP^{-1}$ for all $g \in G$.*

**Remark 0.4.** *We would like to remark that the concept of a linear representation of $G$ on $V$ is equivalent to $G$ acts on $V$ linearly. More precisely, $G$ acts on the vector space $V$ and the action satisfying*

$$g(v + v') = gv + gv', \quad g(cv) = cg(v)$$

*for all $g \in G$, $v, v' \in V$ and $c \in F$.*

**Definition 0.5.** *Let $\rho, \rho'$ be two representations of $G$ on $V, V'$. They are said to be isomorphic if there is an isomorphism of $\tau : V \to V'$ which is compatible with $\rho$ and $\rho'$. That is,*

$$\tau\rho_s(v) = \rho'_s\tau(v),$$

*for all $s \in G, v \in V$.*

**Example 0.6.** *A representation of degree 1 is a homomorphism $R : G \to \mathbb{C}^*$. Since every element has finite order, $R_g$ is a root of unity. In particular, $|R_g| = 1$.*

**Example 0.7** (Regular representation). *Let $G$ be a finite group of order $g$ and let $V$ be a vector space with basis $\{e_t\}_{t\in G}$. For $s \in G$, let $R_s$ be the linear map of $V$ to $V$ which maps $e_t$ to $e_{st}$. This is called the regular representation of $G$.*

*Note that $e_s = R_s(e_1)$ for all $s \in G$. Hence the image of $e_1 \in V$ form a basis. On the other hand, if $\tau : G \to W$ is a representation with the property that there is a $v \in W$ such that $\{\tau_s(v)\}_{s\in G}$ forms a basis. Then $W$ is isomorphic to the regular representation. This is the case by considering $\tau : V \to W$ with $\tau(e_s) = \rho_s(v)$.*

*More generally, if $G$ acts on a finite set $X$, the one can have a representation similarly on the vector space $V$ with basis $X$. This is called the permutation representation associated to $X$.*

Let $\rho, \rho'$ be two representations of $G$ on $V, V'$, then one can define $\rho \oplus \rho'$, $\rho \otimes \rho'$ naturally. Note that if degree of $\rho$ and $\rho'$ are $d, d'$ respectively, then degree of $\rho \oplus \rho'$ is $d + d'$ and degree of $\rho \otimes \rho'$ is $dd'$.

**Definition 0.8.** *$V$ is irreducible representation if $V$ is not a direct sum of two representation non-trivially.*

One might ask whether a representation is irreducible or not. We threrefore introduce the $G$-invariant subspace as we did in linear algebra.

**Definition 0.9.** *Let $\rho : G \to GL(V)$ be a representation. A vector subspace $W$ of $V$ is said to be a $G$-invariant subspace if $\rho_s(W) \subset W$ for all $s \in G$. It's clear that the restriction of $G$ action on $V$ to $W$ give a representation of $G$ on $W$, which is called the subrepresentation of $V$.*

**Theorem 0.10** (Maschke's Theorem). *Every representation of a finite is a direct sum of irreducible representations.*

*Proof.* It suffices to prove that for any $G$-invariant subspace $W \subset V$. There is a $G$-invariant complement of $W$. By a complement of $W$, we mean a subspace $W'$ such that $W \cap W' = \{e\}$, and $W + W' = V$.

We first pick any complement $W'$. Then $V = W \oplus W'$. Let $p : V \to W$ be the projection. We are going to modify $W'$ to get a $G$-invariant complement.

To this end, we average $p$ over $G$ to get

$$p_0 := \frac{1}{g} \sum_{t\in G} \rho_t p \rho_t^{-1},$$

where $g = |G|$.

One checks that $p_0 : V \to W$ and $p_0(w) = w$ for all $w \in W$. That is, $p_0 : V \to W$ is a projection.

Let $W_0 := ker(p_0)$. We check that $W_0$ is $G$-invariant since

$$\rho_s p_0 \rho_s^{-1} = p_0$$

for all $s \in G$. It follows that if $x \in W_0$, $p_0\rho_s(x) = \rho_s(p_0(x)) = 0$, which shows that $\rho_s(x) \in W_0$.

This proves that the representation on $V$ is isomorphic to $W \oplus W_0$. □

**Remark 0.11.** *A matrix over $\mathbb{C}$ of finite order is diagonalizable. Hence every matrix representation over the field $\mathbb{C}$ is diagonalizable. We therefore assume the field to be the complex number field.*

*Moreover, let $\lambda$ be an eigenvalue of $\rho_s$ for some $s$. Then $|\lambda| = 1$.*

**Definition 0.12.** *Let $\rho : G \to GL(V)$ be a linear representation on the vector space $V$. We define the character as $\chi_\rho := Tr \circ \rho : G \to \mathbb{C}$.*

**Proposition 0.13.** *Let $\chi$ be the character of $\rho : G \to GL(V)$.*

(1) $\chi(1) = n := \mathrm{dim} V$,
(2) $\chi(s^{-1}) = \overline{\chi(s)}$ *for all* $s \in G$,
(3) $\chi(tst^{-1}) = \chi(s)$ *for all* $s, t \in G$.
(4) *if $\chi'$ is the character of another representation $\rho'$, then the character of $\rho \oplus \rho'$ is $\chi + \chi'$.*

One can define a hermitian dot product on characters as

$$< \chi, \chi' > := \frac{1}{g} \sum_{s \in G} \overline{\chi(s)} \chi'(s).$$

The main theorem for character is the following:

**Theorem 0.14.** *Let $G$ be a group of order $g$, and let $\rho_1, \ldots$ represent the isomorphism classes of irreducible representations of $G$. Let $\chi_i$ be the character of $\rho_i$ for each $i$.*

(1) *Orthogonality Relations:*

$$<< \chi_i, \chi_j >= 0 if\ i \neq j,$$

$$< \chi_i, \chi_i >= 1 \text{ for each } i.$$

(2) *The number of isomorphism classes of irreducible representations of $G$ is the same as the number of conjugacy classes of $G$. (denote it by $r$).*
(3) *Let $d_i$ be the degree of $\rho_i$, then $d_i | g$ and*

$$g = \sum_{i=1}^{r} d_i^2.$$

**Example 0.15.** *Consider $G = D_4$. It's clear that $r = 5$. Hence it's only possible to have $d_1 = 2, d_2 = \ldots = d_5 = 1$.*