# Advanced Algebra I

## More examples

Let $\rho : G \to GL(V)$ be a representation. It's clear that $\ker(\rho)$ is a normal subgroup of $G$. One has

$$ker(\rho) = ker(\chi),$$

where $ker(\chi) := \{s \in G | \chi(s) = \chi(1)\}$. Therefore, by looking at the character table, it's possible to obtain some information on normal subgroups and hence on the structure of the group. We illustrate the following example.

**Example 0.1.** *Let $G$ be a finite group with the following character table. We would like to determine the structure of the group $G$. Where $\omega = \frac{-1+\sqrt{3}i}{2}$ is the third root of unity.*

| | (1) 1 | (6) $a$ | (7) $b$ | (7) $c$ | (7) $d$ | (7) $e$ | (7) $f$ |
|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $\omega$ | $\bar{\omega}$ | $\omega$ | $\bar{\omega}$ |
| $\chi_3$ | 1 | 1 | 1 | $\bar{\omega}$ | $\omega$ | $\bar{\omega}$ | $\omega$ |
| $\chi_4$ | 1 | 1 | $-1$ | $-\omega$ | $-\bar{\omega}$ | $\omega$ | $\bar{\omega}$ |
| $\chi_5$ | 1 | 1 | $-1$ | $-\bar{\omega}$ | $-\omega$ | $\bar{\omega}$ | $\omega$ |
| $\chi_6$ | 1 | 1 | $-1$ | $-1$ | $-1$ | 1 | 1 |
| $\chi_7$ | 6 | $-1$ | 0 | 0 | 0 | 0 | 0 |

(1) *The group $G$ has order* 42.

(2) *We have normal subgroups $N_i := ker(\chi_i)$ for $i = 2, 4, 6$. Note that $|N_2| = 14, |N_4| = 7, |N_6| = 21$.*

*Moreover, If $N_2$ is abelian, then irreducible representation of $G$ has degree $\leq \frac{|G|}{|N_2|} = 3$ which is impossible. Hence $N_2$ is non-abelian. So is $N_3$.*

(3) $o(a) = 7, o(c) = 6$. *To see these, note that*

$$6 = |c_a| = \frac{|G|}{|C_G(a)|},$$

*where $c_a$ denotes the conjugacy class of $a$ and $C_G(a)$ denotes the centralizer of $a$ in $G$. It follows that $< a >\subset C_G(a)$ has order* 7.

*Similarly, one has $< c >\subset C_G(c)$ hence $o(c)|6$. On the other hand, $6 = o(\rho_4(c))|o(c)$. It's clear that $o(c) = 6$.*

(4) $< a >= N_4 \lhd G$. *Thus $G =< a, c | a^7 = c^6 = e, cac^{-1} = a^k >$. One must have $k^6 \equiv 1 \pmod 7$. That is, $k \equiv 1, .., 6 \pmod 7$.*

(5) *we finally claim that $k \equiv 3, 5$. If $k \equiv 1$, then $G$ is abelian hence $\cong \mathbb{Z}_{42}$ which is impossible. If $k \equiv 2$, then $c^3ac^{-3} = a$. Thus*

$C_G(c^3) \supseteq < c^3, a >$. *It follows that* $|c_{c^3}| < 6$ *which is impossible. The argument is similar for* $k \equiv 4, 6$.

(6) *We remark that the groups* $G_3 :=< a, c|a^7 = c^6 = e, cac^{-1} = a^3 >$ *and* $G_5 :=< a, c|a^7 = c^6 = e, cac^{-1} = a^5 >$ *are isomorphic by* $G_5 \to G_3$ *which sends* $a \mapsto a, c \mapsto c^5$.

Another important example of constructing representation is the *permutation representation*. Let $G$ be a group acting on a set $S$. We consider a vector space $\mathbb{C}[S]$ with basis $\{e_x\}_{x \in S}$. Via the group action $\rho : G \times S \to S$, one can produce an action $\widetilde{\rho} : G \times \mathbb{C}[S] \to \mathbb{C}[S]$ by $\widetilde{\rho}_s(e_x) = e_{sx}$. This clearly gives a representation of $G$ on $\mathbb{C}[S]$. one notice that the matrix for such permutation representation is a permutation matrix. The diagonal entries is 1 if and only if $e_x = e_{sx}$. Therefore, one has

$$\chi_{\widetilde{\rho}}(s) = |\{x \in S | sx = x\}|.$$

Moreover, let $e = \sum_{x\ inS} e_x$. It's clear that $\widetilde{\rho}_s(e) = e$. Hence $V = \mathbb{C}e \subset \mathbb{C}[G]$ is an invariant subspace such that $\widetilde{\rho}|_V = 1$. Thus one has a decomposition

$$\widetilde{\rho} = 1 \oplus \rho'.$$

**Example 0.2.** *Let* $G = S_3$. *Then* $G$ *acts on* $S = \{1, 2, 3\}$ *naturally. We have two representation of degree* 1. *Together with* $\widetilde{\rho} = 1 \oplus \rho'$, *one has the following table:*

| | (1) 1 | (3) (12) | (2) (123) |
|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 |
| $\chi_{\widetilde{\rho}}$ | 3 | 1 | 0 |
| $\chi_{\rho'}$ | 2 | 0 | $-1$ |

*It's easy (by orthogonality) to see that* $\chi_{\rho'}$ *is irreducible. And hence we have completes the table of irreducible characters.*

**Example 0.3.** *Let* $G$ *be a non-abelian group of order* 21. *Then it's clear that* $G =< a, b|a^7 = b^3 = e, bab^{-1} = a^2 >$.

(1) *By direct computations, one finds that there are conjugacy classes* $c_1 = \{e\}, c_2 = \{a, a^2, a^4\}, c_3 = \{a^3, a^5, a^6\}, c_4 = \{b, ab, ..., a^6b\}$,$c_5 = \{b^2, ab^2, ..., a^6b^2\}$.

(2) $N =< a >$ *is clearly a normal subgroup. The irreducible representations of* $G/N \cong \mathbb{Z}_3$ *gives* 3 *irreducible representation of degree* 1. *Where* $\rho_1$ *is the trivial one,* $\rho_2(a^ib^j) = \omega^j$, $\rho_3(a^ib^j) = \omega^{2j}$.

(3) *Let* $\rho_4, \rho_5$ *be the remaining irreducible representations (of degree* $d_4, d_5$ *respectively). By* $21 = 3 + d_4^2 + d_5^2$, *one has* $d_4 = d_5 = 3$.

*We have now*

| | (1) | (3) | (3) | (7) | (7) |
|---|---|---|---|---|---|
| | $e$ | $a$ | $a^3$ | $b$ | $b^2$ |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | 1 | 1 | $\omega^2$ | $\omega$ |
| $\chi_4$ | 3 | | | | |
| $\chi_5$ | 3 | | | | |

(4) *Let $S$ be the set of Sylow 3-subgroups. $|S| = 1$ or 7. One sees that $|S| = 7$ otherwise let $P$ be the only Sylow 3-subgroup which is normal, it follows that $G = NP$ and hence $G = N \oplus P \cong \mathbb{Z}_{21}$.*

(5) *$G$ acting on $S$ gives a permutation representation $\widetilde{\rho}$ of degree 7. By computation, $\chi_{\rho'} = 6, -1, -1, 0, 0$ on $e, a, a^3, b, b^2$ respectively. One can checks that $< \chi_{\rho'}, \chi_i >= 0$ for $i = 1, 2, 3$. Thus*

$$\chi_{\rho'} = n_4\chi_4 + n_5\chi_5,$$

*with $n_4, n_5 \geq 0$, $n_4 + n_5 = 2$. However, $(n_4, n_5)$ can't be $(2, 0)$ or $(0, 2)$ cause otherwise $\chi_4(a) = \frac{-1}{2} \notin \mathcal{A}$ (or $\chi_5(a)$). Hence one has*

$$\chi_{\rho'} = \chi_4 + \chi_5.$$

(6) *By using the orthogonal properties, one can solve for $\chi_4, \chi_5$ and we obtain the following complete table:*

| | (1) | (3) | (3) | (7) | (7) |
|---|---|---|---|---|---|
| | $e$ | $a$ | $a^3$ | $b$ | $b^2$ |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | 1 | 1 | $\omega^2$ | $\omega$ |
| $\chi_4$ | 3 | $\zeta$ | $\bar{\zeta}$ | 0 | 0 |
| $\chi_5$ | 3 | $\bar{\zeta}$ | $\zeta$ | 0 | 0 |

*where $\zeta = \frac{-1+\sqrt{7}i}{2}$.*