Advanced Algebra I

Sep. 19,20, 2003 (Fri. Sat.)

1. Set Theory

We recall some set theory that will be frequently used in the sequel or that is not covered in the basic college course.

1.1. **Zorn's Lemma.**

Definition 1.1. *A set S is said to be partially ordered if there is a relation $\leq$ such that*

(1) $x \leq x$
(2) *if* $x \leq y$ *and* $y \leq x$*, then* $x = y$.
(3) *if* $x \leq y$ *and* $y \leq z$ *then* $x \leq z$.

We usually call a partially ordered set to be a POSET.

Definition 1.2. *A pair of elements is said to be comparable if either $x \leq y$ or $y \leq x$. A set is said to be totally ordered if every pair is comparable.*

We also need the following definition:

Definition 1.3. *A maximal element of an poset S is an element $m \in S$ such that if $m \leq x$ then $m = x$.*

Foe a given subset $T \subset S$, an upper bound of T is an element $b \in S$ such that $x \leq b$ for all $x \in T$.

One has

Theorem 1.4 (Zorn's lemma)**.** *Let S be a non-empty poset. If every non-empty totally ordered subset (usually called a "chain") has an upper bound, then there exists a maximal element in S.*

Example 1.5. *Let R be a $\neq 0$ commutative ring. One can prove that there exists a maximal ideal by using Zorn's lemma. The proof goes as following: Let $S = \{I \lhd R | I \neq R\}$ equipped with the $\subset$ as the partial ordering. $S \neq \emptyset$ because $0 \in S$. For a chain $\{I_j\}_{j \in J}$, one has a upper bound $I = \cup I_j$. Then we are done by Zorn's lemma.*

1.2. **cardinality.**

In order to compare the "zise of sets", we introduce the cardinality.

Definition 1.6. *Two sets A, B are said to have the same cardinality if there is a bijection between them, denoted $|A| = |B|$. And we said $|A| \leq |B|$ if there is a injection from A to B.*

It's easy to see that the cardinality has the properties that $|A| \leq |A|$ and if $|A| \leq |B|, |B| \leq |C|$, then $|A| \leq |C|$. So It's likely that the "cardinality are partially ordered" or even totally ordered.

Lemma 1.7. *Given two set A, B, either $|A| \leq |B|$ or $|B| \leq |A|$.*

Sketch. Consider

$$S = \{(C, f)|C \subset A, f : C \to B \text{ is an injection}\}.$$

Apply Zorn's lemma to S, one has an maximal element (D, g), then one claim that either $D = A$ or $im(g) = B$. □

Theorem 1.8 (Schroeder-Bernstein)**.** *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

sketch. Let f, g be the injection from A, B to B, A respectively. One needs to construct a bijection by using f and g. Some parts of A use f and some parts not. So we consider the partition

$$A_1 := \{a \in A|a \text{has parentless ancestor in } A\},$$
$$A_2 := \{a \in A|a \text{has parentless ancestor in } B\},$$
$$A_3 := \{a \in A|a \text{has infinite ancestor}\}.$$

And so does B.

Then we claim that f restricted to A_1, A_3 are bijections to B_1, B_3. And g restricted to B_2, B_3 are bijections to A_2, A_3. So the desired bijection can be constructed. □

We need some more properties of cardinality. If $|A| = |\{1, .., n\}$, then we write $|A| = n$. And if $|A| = |\mathbb{N}$ then we write $|A| = \aleph_0$.

Proposition 1.9. *If A is infinite, then $\aleph_0 \leq |A|$.*

Proof. By Axiom of Choice. □

Definition 1.10.

$$|A| + |B| := |A \amalg B|,$$
$$|A| \cdot |B| := |A \times B|.$$

We have the following properties:

Proposition 1.11. (1) *If $|A|$ is infinite and $|B|$ is finite, then $|A+B| = |A|$.*
(2) *If $|B| \leq |A|$ and $|A|$ is infinite, then $|A + B| = |A|$.*
(3) *If $|B| \leq |A|$ and $|A|$ is infinite, then $|A \times B| = |A|$.*

Proof. For (1), take an countable subset A_0 in A, one sees that $|A_0| = |A_0| + |B|$ by shifting. Then we are done.

For (2), It's enough to see that $|A + A| \leq |A|$. Pick an maximal subset $X \subset A$ having the property that $|X + X| \leq |X|$ (by Zorn's Lemma). One claim that $A - X$ is finite, and then we are done by (1).

For (3), we leave it as an exercise to the readers. □