Advanced Algebra I

Homework 7

due on Nov.7, 2003

Part A.

(1) Let $\rho : G \to GL(V)$ be a representation on the complex vector space V of dimension d. And χ is the character of ρ.

(a) Show that $|\chi(s)| \leq d$ for all $s \in G$. And equality holds if and only if $\rho_s = \lambda I$.

(b) We can define $ker(\chi) = \{s \in G | \chi(s) = \chi(1)\}$. Show that $\ker(\chi) = ker(\rho)$ and it's a normal subgroup of G.

(c) Let $\lambda_1, ..., \lambda_n$ be roots of unity, and let $a = \frac{1}{n}\sum_{i=1}^{n} \lambda_i$. Prove that if a is an algebraic integer, then either $a = 0$ or $a = \lambda_1 = ..., = \lambda_n$.

(2) Let $\rho : G \to GL(V)$ be an irreducible representation of G of degree > 1. Given $v \in V$, we consider $w := \sum_{s \in G} \rho_s(v)$. Prove that w is G-invariant. And prove that $w = 0$.

Part B.

(1) The following is part of a character table with 2 characters missing:

	(1)	(3)	(3)	(7)	(7)
	1	a	b	c	d
χ_1	1	1	1	ω	$\bar{\omega}$
χ_2	3	γ	$\bar{\gamma}$	0	0
χ_3	3	$\bar{\gamma}$	γ	0	0

Where $1, a, b, c, d$ are representatives of conjugacy classes $c_1, .., c_5$. And the conjugacy classes contains $1, 3, 3, 7, 7$ elements respectively. And $\omega = \frac{-1+\sqrt{3}i}{2}$ and $\gamma = \frac{-1+\sqrt{7}i}{2}$.

(a) Determine the remaining characters.

(b) Determine the order of a, b, c, d respectively.

(c) Show that the group has a non-trivial normal subgroup.

(d) Determine the group by generators and relations.

(e) What is the commutator of this group? Describe it in terms of union of conjugacy classes.

(2) Consider the group algebra $\mathbb{C}[S_3]$. Find all elements $u \in \mathbb{C}[S_3]$ such that $u^2 = 1$. (Hint: it might be helpful to consider $\tilde{\rho} : \mathbb{C}[S_3] \to \prod M_{d_i}(\mathbb{C})$.)