Basic Algebra (Solutions)

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Exercises (§1.6, pp.50–51)

1. Write (456)(567)(671)(123)(234)(345) as a product of disjoint cycles.

Ans. (127).

2. Show that if $n \geq 3$ then A_n is generated by the 3-cycles (abc).

Proof. Since A_n is generated by the permutations which is a product of two transpositions, hence it suffices to show that $(ij)(kl)$ is generated by 3-cycles. ¿From the identities $(ij)(ik) = (ikj)$ and $(ij)(kl) = (ijk)(jkl)$, the assertion follows. □

3. Determine the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$

Ans.

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix} = \begin{cases} (1n)(2, n-1)\cdots(\frac{n}{2}, \frac{n}{2}+1), & \text{if } n \text{ is even} \\ (1n)(2, n-1)\cdots(\frac{n-1}{2}, \frac{n+3}{2}), & \text{if } n \text{ is odd} \end{cases}$$

$$\text{sgn } \alpha = \begin{cases} 1, & \text{if } n = 0 \text{ or } 1 \pmod 4 \\ -1, & \text{if n=2 or 3 (mod 4).} \end{cases}$$

4. Show that if α is any permutation then

$$\alpha(i_1 i_2 \cdots i_r)\alpha^{-1} = (\alpha(i_1)\alpha(i_2)\cdots\alpha(i_r)).$$

Proof. Let j be any number in $\{1, 2, \ldots, n\}$. (1) if $j = \alpha(i_k)$ for some k, then $[\alpha(i_1 i_2 \cdots i_r)\alpha^{-1}](j) = [\alpha(i_1 i_2 \cdots i_r)](i_k) = \alpha(i_{k+1})$. On the other hand $(\alpha(i_1)\alpha(i_2)\cdots\alpha(i_r))(j) = \alpha(i_{k+1})$, where $i_{r+1} = i_1$. Hence the result. (2) If $j \neq \alpha(i_k)$ for all k, then $\alpha^{-1}(j)$ is fixed by the permutation $(i_1 \cdots i_r)$. Hence $[\alpha(i_1 i_2 \cdots i_r)\alpha^{-1}](j) = (\alpha\alpha^{-1})(j) = j$ and $(\alpha(i_1)\cdots\alpha(i_r))(j) = j$. The result follows. □

5. Show that S_n is generated by the $n-1$ transpositions (12), (13), $\ldots$, $(1n)$ and also by the $n-1$ transpositions (12), (23), $\ldots$, $(n-1, n)$.

Proof. (1) From the identity $(a, b) = (1a)(1b)(1a)$, we see that any transposition is generated by $(1i)$'s. Hence S_n is generated by $(1i)$'s.

(2) It suffices to show that for any transposition $(1j)$ can be written as a product of $(i, i+1)$'s. This is clear from $(1j) = (j-1, j)(j-2, j-1)\cdots(23)(12)(23)\cdots(j-2, j-1)(j-1, j)$. □