Basic Algebra (Solutions)

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Exercises (§1.10, p.65)

1. Show that $\mathbb{Z}l \cap \mathbb{Z}k = \mathbb{Z}[l,k]$ and $\mathbb{Z}l + \mathbb{Z}k = \{a+b | a \in \mathbb{Z}l, b \in \mathbb{Z}k\} = \mathbb{Z}(l,k)$.

Proof. (1) $\mathbb{Z}l \cap \mathbb{Z}k = \mathbb{Z}[l,k]$:
Since $a \in \mathbb{Z}l \cap \mathbb{Z}k \Leftrightarrow l|a$ and $k|a \Leftrightarrow [l,k]|a \Leftrightarrow a \in \mathbb{Z}[l,k]$.

(2) $\mathbb{Z}l + \mathbb{Z}k = \mathbb{Z}(l,k)$.
Since a g.c.d. $(l,k) = xl + yk$ for some $x, y \in \mathbb{Z}$ (see p.23), hence $\mathbb{Z}(l,k) \subset \mathbb{Z}l + \mathbb{Z}k$. On the other hand, $(l,k)|l$ and $(l,k)|k$ imply that $\mathbb{Z}l \subset \mathbb{Z}(l,k)$ and $\mathbb{Z}k \subset \mathbb{Z}(l,k)$. Hence the result. □

2. Let $\{H_\alpha\}$ be a collection of subgroups containing the normal subgroup K. Show that $\cap(H_\alpha/K) = (\cap H_\alpha)/K$.

Proof. We just need to prove that: if $xK \in \cap(H_\alpha/K)$ then $x \in \cap H_\alpha$.

For any α, since $xK \in H_\alpha/K$, so $xK = x_\alpha K$ for some $x_\alpha \in H_\alpha$. Then $xx_\alpha^{-1} \in K \subset H_\alpha$ and $x \in H_\alpha$. Hence the result.