Basic Algebra (Solutions)

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Exercises (§1.1, p.30)

1. Let S be a set and define a product in S by $ab = b$. Show that S is a semigroup. Under what condition does S contain a unit?

Ans. S contains a unit if and only if S is a singleton. The verification is left for the reader.

2. Let $M = \mathbb{Z} \times \mathbb{Z}$ the set of pairs of integers (x_1, x_2). Define $(x_1, x_2)(y_1, y_2) = (x_1y_1 + 2x_2y_2, x_1y_2 + x_2y_1)$, $1 = (1, 0)$. Show that this defines a monoid. (Observe that the commutative law of multiplication holds.) Show that if $(x_1, x_2) \neq (0, 0)$ then the cancellation law will hold for (x_1, x_2), that is, $(x_1, x_2)(y_1, y_2) = (x_1, x_2)(z_1, z_2) \Rightarrow (y_1, y_2) = (z_1, z_2)$.

Sol. The cancellation law: Suppose $(x_1, x_2)(y_1, y_2) = (x_1, x_2)(z_1, z_2)$, $(x_1, x_2) \neq (0, 0)$. Then $(x_1y_1 + 2x_2y_2, x_1y_2 + x_2y_1) = (x_1z_1 + 2x_2z_2, x_1z_2 + x_2z_1)$. Comparing both components, we have

$$\begin{cases} x_1(y_1 - z_1) + 2x_2(y_2 - z_2) = 0 \\ x_2(y_1 - z_1) + x_1(y_2 - z_2) = 0 \end{cases}$$

Since the determinant $\begin{vmatrix} x_1 & 2x_2 \\ x_2 & x_1 \end{vmatrix} = x_1^2 - 2x_2^2 \neq 0$ (note that $x_1, x_2 \in \mathbb{Z}$), hence the system of linear equations has trivial solution $y_1 - z_1 = 0$ and $y_2 - z_2 = 0$, that is, $(y_1, y_2) = (z_1, z_2)$.

3. A machine accepts eight-letter words (defined to be any sequence of eight letters of the alphabet, possible meaningless), and prints an eight-letter word consisting of the first five letters of the first word followed by the last three letters of the second word. Show that the set of eight-letter words with this composition is a semigroup. What if the machine prints the last four letters of the first word followed by the first four of the second? Is either of these systems a monoid?

Ans. Let $(a_1, \ldots, a_8)$ and $(b_1, \ldots, b_8)$ be two words. If we define the composition to be $(a_1, \ldots, a_8)(b_1, \ldots, b_8) = (a_1, \ldots, a_5, b_6, b_7, b_8)$, then $\{(a_1, a_2, \ldots, a_8)(b_1, \ldots, b_8)\}(c_1, \ldots, c_8) = (a_1, \ldots, a_5, c_6, c_7, c_8)$ and $(a_1, \ldots, a_8)\{(b_1, \ldots, b_8)(c_1, \ldots, c_8)\} = (a_1, \ldots, a_5, c_6, c_7, c_8)$. Thus it is a semigroup. But it is not a monid. If we define $(a_1, \ldots, a_8)(b_1, \ldots, b_8) = (a_5, a_6, a_7, a_8, b_1, b_2, b_3, b_4)$, then $\{(a_1, \ldots, a_8)(b_1, \ldots, b_8)\}(c_1, \ldots, c_8) =$

$(b_1, \ldots, b_4, c_1, \ldots, c_4)$ and $(a_1, \ldots, a_8)\{(b_1, \ldots, b_8)(c_1, \ldots, c_8)\} = (a_5, \ldots, a_8, b_5, \ldots, b_8)$. Thus it is not a semigroup and hence not a monoid.

4. Let $(M, p, 1)$ be a monoid and let $m \in M$. Define a new product p_m in M by $p_m(a, b) = amb$. Show that this defines a semigroup. Under what condition on m do we have a unit relative to p_m?

Ans. We call an element $x \in M$ invertible if there is some element $y \in M$ such that $xy = 1 = yx$ (See §1.2); we denote y by x^{-1}. Then M has a unit relative to p_m if and only if m is invertible. In that situation, the unit is m^{-1}.

5. Let S be a semigroup, u an element not in S. Form $M = S \cup \{u\}$ and extend the product in S to a binary product in M by defining $ua = a = au$ for all $a \in M$. Show that M is a monoid.

Proof. Omitted. $\square$