Basic Algebra (Solutions)

by Huah Chu

Exercises (§0.5, p.21)

1. Show that $x \geq y \Leftrightarrow -x \leq -y$.

Proof. Let $x = \overline{(a,b)}$ and $y = \overline{(c,d)}$. Then $x \geq y \Leftrightarrow a + d \geq b + c \Leftrightarrow d + a \geq c + b \Leftrightarrow \overline{(d,c)} \geq \overline{(b,a)} \Leftrightarrow -y \geq -x$. □

2. Prove that any non-vacuous set S of integers which is bounded below (above), in the sense that there exists an integer $b(B)$ such that $b \leq s$ $(B \geq s)$, $s \in S$, has a least (greatest) element.

Proof. We prove the case of bounded below only. Let b be the lower bound of S. Consider the set $S - b = \{s - b | s \in S\}$. Clearly $S - b \subset \mathbb{N}$. By the well-ordered property, $S - b$ has a least element $s_0 - b$. It follows that s_0 is a least element of S. □

3. Define $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Prove that $|xy| = |x||y|$ and $|x + y| \leq |x| + |y|$.

Proof. (0) We first prove that $(-x)y = -(xy) = x(-y)$, $-(x + y) = (-x) + (-y)$: Let $x = \overline{(a,b)}$, $y = \overline{(c,d)}$. Then $(-x)y = \overline{(b,a)}\overline{(c,d)} = \overline{(bc + ad, bd + ac)} = -\overline{(ac + bd, ad + bc)} = -(xy)$. The other equalities can be proved similarly.

(1) (i) If $x \geq 0$ and $y \geq 0$, then $|xy| = xy = |x||y|$.

(ii) If $0 \geq x$ and $y \geq 0$, then $0 = 0y \geq xy$ by OM'. Thus $|xy| = -(xy) = (-x)y = |x||y|$. The case of $x \geq 0$, $0 \leq y$ is similar.

(iii) If $x \leq 0$, $y \leq 0$, then $-(xy) = (-x)y \leq 0$ (by (ii)) and $xy \geq 0$. Thus $|xy| = xy = -((-x)y) = (-x)(-y) = |x||y|$.

(2) $|x + y| \leq |x| + |y|$:

(i) If $x \geq 0$ and $y \geq 0$, then $x + y \geq 0$. Hence $|x + y| = x + y = |x| + |y|$.

(ii) If $x \geq 0$ and $y \leq 0$, then $|x + y|$ is equal to $x + y$ or $-(x + y)$. Since $y \leq 0$, $y \leq -y$, hence $x + y \leq x + (-y)$. On the other hand, $-x \leq x$, hence $-(x + y) = (-x) + (-y) \leq x + (-y)$. In any case, $|x + y| \leq x + (-y) = |x| + |y|$.

(iii) If $x \leq 0$ and $y \geq 0$, we can prove the inequality as (ii).

(iv) If $x \leq 0$ and $y \leq 0$, then $x + y \leq 0$. Thus $|x + y| = -(x + y) = (-x) + (-y) = |x| + |y|$. □