Basic Algebra (Solutions)

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Exercises (§0.4, pp.18–19)

1. Prove that if $a \geq b$ and $c \geq d$ then $a + c \geq b + d$ and $ac \geq bd$.

Proof. (1) $a \geq b$ and $c \geq d$, then $a + c \geq b + c$ and $c + b \geq d + b$ by OA. By the commutative law $A2$, we have $b + d \geq b + d$. Hence $a + c \geq b + d$ by $O2$.

(2) (Omitted.)

2. Prove the following extension of the first principle of induction: Let $s \in \mathbb{N}$ and assume that for every $n \geq s$ we have a statement $E(n)$. Suppose $E(s)$ holds, and if $E(r)$ holds for some $r \geq s$, then $E(r^+)$ holds. Then $E(n)$ is true for all $n \geq s$. State and prove the analogous extension of the second principle of induction.

Proof. (1) The extension of the first principle of induction: We shall show that the subset $F = \{n \in \mathbb{N} | n > s \text{ and } E(n) \text{ is false}\}$ is vacuous. Suppose not. Then by the well-ordering principle, F contains a least element ℓ. Then $E(\ell - 1)$ is true ($\ell - 1 \geq s$). This implies that $E((\ell - 1)^+)$ is true by the hypothese. A contradiction.

(2) The extension of the second principle of induction: Let $s \in \mathbb{N}$ and assume that for every $n \geq s$, we have a statement $E(n)$. Suppose for any particular $r > s$, $E(r)$ is true if $E(t)$ is true for all $s \leq t < r$. Then $E(n)$ is true for all n. The proof is analogue to that of (1) and we left to the reader. □

3. Prove by induction that if c is a real number ≥ -1 and $n \in \mathbb{N}$ then $(1+c)^n \geq 1+nc$.

Proof. (Omitted.) □

4. (Henkin.) Let $N = \{0, 1\}$ and define $0^+ = 1$, $1^+ = 1$. Show that N satisfies Peano's axioms 1 and 3 but not 2. Let ϕ be the map of N into N such that $\phi(0) = 1$ and $\phi(1) = 0$. Show that the recursion theorem breaks down for N and this ϕ, that is, there exists no map f of N into itself satisfying $f(0) = 0$, $f(n^+) = \phi(f(n))$.

Proof. The first statement is trivial. We prove the second. Suppose that we have a map $f : N \to N$ satisfying $f(0) = 0$, $f(n^+) = \phi(f(n))$. Then

$$f(1) = f(0^+) = \phi(f(0)) = \phi(0) = 1.$$

But

$$f(1) = f(1^+) = \phi(f(1)) = \phi(1) = 0.$$

It leads to a contradiction. $\square$

5. Prove $A1$ and $M2$.

Proof. (1) $A1$. $(x+y)+z = x+(y+z)$: Fix y and z, we shall prove this equality by induction.

(i) $x = 0$; $(0+y)+z = y+z = 0+(y+z)$.

(ii) Suppose it holds for x, then

$$\begin{aligned}(x^+ + y) + z &= (x+y)^+ + z = ((x+y)+z)^+ \\ &= (x+(y+z))^+ = x^+ + (y+z).\end{aligned}$$

Therefore the assertion holds for all x.

(2) Before proving $M2$, we first show that $xy^+ = xy + x$: Fix y, we prove this statement hold for all $x \in \mathbb{N}$ by induction.

(i) As $x = 0$, $0y^+ = 0 = 0 \cdot y + 0$.

(ii) Suppose it holds x. Then $x^+y^+ = xy^+ + y^+ = (xy+x) + y^+ = xy + (x + y^+) = xy + (y^+ + x) = xy + (y+x)^+ = xy + (x+y)^+ = xy + (x^+ + y) = xy + (y + x^+) = (xy+y) + x^+ = x^+y + x^+$.

The assertion is proved.

(3) $M2$. $xy = yx$: Fix y, we prove it by induction on x.

(i) $x = 0$. Then $0y = 0$ by definition. Next, we show that $y \cdot 0 = 0$. Because of $0 \cdot 0 = 0$ and $y^+ \cdot 0 = y \cdot 0 + 0 = 0 + 0 = 0$, the assertion holds. Thus $0 \cdot y = y \cdot 0$.

(ii) Suppose $xy = yx$. Then $x^+y = xy + y = yx + y = yx^+$ by (2). Hence $xy = yx$ holds for all x. $\square$