Basic Algebra (Solutions)

by Huah Chu

Exercises (§0.2, p.10)

1. Let $S = \{1, 2, \ldots\}$. Give an example of two maps α, β of S into S that $\alpha\beta = 1_S$ but $\beta\alpha \neq 1_S$. Can this happen if α is bijective?

Sol. (1) Let $\alpha : S \to S$ be defined by $\alpha(n) = \begin{cases} n-1 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$ and $\beta : S \to S$ be defined by $\beta(n) = n + 1$ for $n \in S$. Then $\alpha\beta = 1_S$ and $\beta\alpha \neq 1_S$.

(2) If α is bijective and $\alpha\beta = 1_S$, then $\beta\alpha = 1_S$.

Proof. If α is bijective, then the inverse α^{-1} exists. Hence

$$\begin{aligned} \beta\alpha &= 1_S(\beta\alpha) = (\alpha^{-1}\alpha)(\beta\alpha) = \alpha^{-1}(\alpha\beta)\alpha \\ &= \alpha^{-1}(1_S\alpha) = \alpha^{-1}\alpha = 1_S. \end{aligned}$$

□

2. Show that $S \xrightarrow{\alpha} T$ is injective if and only if there is a map $T \xrightarrow{\beta} S$ such that $\beta\alpha = 1_S$, surjective if and only if there is a map $T \xrightarrow{\beta} S$ such that $\alpha\beta = 1_T$. In both cases investigate the assertion: If β is unique then α is bijective.

Proof. (1) Let $S \xrightarrow{\alpha} T$ be injective. Note that

$$T = \alpha(S) \cup \{T - \alpha(S)\}.$$

Let a be any element in S. Define $\beta : T \to S$ as follows:

$$\beta(t) = \begin{cases} s, & \text{if } t = \alpha(s) \text{ for some } s \in S. \\ a, & \text{if } t \notin \alpha(S). \end{cases}$$

It is easy to verify that $\beta(t)$ is well-defined when $t \in \alpha(S)$. Moreover $\beta\alpha = 1_S$.

Conversely, suppose $\beta\alpha = 1_S$ and $\alpha(s_1) = \alpha(s_2)$ for $s_i \in S$. Then $s_1 = (\beta\alpha)(s_1) = \beta(\alpha(s_1)) = \beta(\alpha(s_2)) = (\beta\alpha)(s_2) = s_2$. Hence α is injective.

(2) Let $S \xrightarrow{\alpha} T$ be surjective. For any $t \in T$, the subset $\{\alpha(s) = t | s \in S\}$ is not empty. We choose any element (s_t, t) contained in this subset. (Here we use "axiom

of choice".) Define $\beta : T \to S$ by $\beta(t) = s_t$ for all $t \in T$. Again it is routine to check that $\alpha\beta = 1_T$. Conversely, suppose $\alpha\beta = 1_T$. For any $t \in T$, let $s = \beta(t)$. Then $\alpha(S) = \alpha(\beta(t)) = 1_T(t) = t$. Hence α is surjective.

(3) (i) Suppose $\alpha : S \to T$ is injective but not bijective. If S is not a singleton (a set with only one element), we shall show that β is not unique: $T \neq \alpha(S)$, otherwise α is bijective. Now choose $a_1, a_2 \in S$ with $a_1 \neq a_2$. Define $\beta_1, \beta_2 : T \to S$ by $\beta_1(\alpha(s)) = s = \beta_2(\alpha(s))$ for all $s \in S$ and $\beta_1(t) = a_1$, $\beta_2(t) = a_2$ for all $t \in T - \alpha(S)$. Then $\beta_1 \neq \beta_2$ but $\beta_1\alpha = \beta_2\alpha = 1_S$.

(ii) Suppose $\alpha : S \to T$ is surjective but not bijective. We shall show that the choice of β is not unique. □

3. Show that $S \xrightarrow{\alpha} T$ is surjective if and only if there exist no maps β_1, β_2 of T into a set U such that $\beta_1 \neq \beta_2$ but $\beta_1\alpha = \beta_2\alpha$. Show that α is injective if and only if there exist no maps y_1, y_2 of a set U into S such that $y_1 \neq y_2$ but $\alpha y_1 = \alpha y_2$.

Remark. To prove that first statement, we need the condition of S being not a singleton.

Proof. (1) Let $S \xrightarrow{\alpha} T$ be surjective, $\beta_1, \beta_2 : T \to U$ with $\beta_1\alpha = \beta_2\alpha$. We shall show that $\beta_1 = \beta_2$. For any $t \in T$, find $s \in S$ such that $\alpha(s) = t$. Then $\beta_1(t) = \beta_1(\alpha(s)) = (\beta_1\alpha)(s) = (\beta_2\alpha)(s) = \beta_2(\alpha(s)) = \beta_2(t)$. Hence the result. (We can prove the assertion using the result in exercise 2 also.)

Suppose $S \xrightarrow{\alpha} T$ is not surjective. Choose two distinct elements $a, b \in S$. We define $\beta_1 : T \to S$ as follows: $\beta_1(t) = a$ for all $t \in T$; $\beta_2(t) = a$ for all $t \in \alpha(S)$ and $\beta_2(t) = b$ for all $t \notin \alpha(S)$. Then $\beta_1 \neq \beta_2$ and $\beta_1\alpha = \beta_2\alpha$ since $\beta_i\alpha(s) = a$ for all $s \in S$.

(2) Let $S \xrightarrow{\alpha} T$ be injective. There exists $T \xrightarrow{\beta} S$ such that $\beta\alpha = 1_S$ by exercise 2. Suppose $\alpha y_1 = \alpha y_2$. Then $y_1 = 1_S y_1 = \beta\alpha y_1 = \beta\alpha y_2 = y_2$.

Suppose $S \xrightarrow{\alpha} T$ is not injective, that is, there exist $a \neq b \in S$, $c \in T$ such that $\alpha(a) = \alpha(b) = c$. We define $y_i : T \to S$ as follows: $y_1(t) = a$, $y_2(t) = b$, for all $t \in T$. Then $y_1 \neq y_2$ and $\alpha y_1 = \alpha y_2$. Since $\alpha y_i(t) = c$, for all $t \in T$. □

4. Let $S \xrightarrow{\alpha} T$ and let A and B be subsets of S. Show that $\alpha(A \cup B) = \alpha(A) \cup \alpha(B)$ and $\alpha(A \cap B) \subset \alpha(A) \cap \alpha(B)$. Give an example to show that $\alpha(A \cap B)$ need not coincide with $\alpha(A) \cap \alpha(B)$.

Proof. (1) $\alpha(A \cup B) = \{\alpha(s) | s \in A \cup B\} = \{\alpha(s) | s \in A \text{ or } s \in B\} = \{\alpha(s) | s | inA\} \cup \{\alpha(s) | s \in B\} = \alpha(A) \cup \alpha(B)$.

(2) Since $A \cap B \subset A$ and $A \cap B \subset B$, $\alpha(A \cap B) \subset \alpha(A)$ and $\alpha(A \cap B) \subset \alpha(B)$. Thus $\alpha(A \cap B) \subset \alpha(A) \cap \alpha(B)$.

(3) Counterexample:

Let $S = \{1, 2, 3\}$, $A = \{1, 2\}$, $B = \{2, 3\}$, and $T = \{a, b\}$. Define $S \xrightarrow{\alpha} T$ by $\alpha(1) = a$, $\alpha(2) = b$, $\alpha(3) = a$. Then $\alpha(A \cap B) = \{b\}$, $\alpha(A) \cap \alpha(B) = \{a, b\}$. $\square$

5. Let $S \xrightarrow{\alpha} T$, and let A be a subset of S. Let the complement of A in S, that is, the set of elements of S not contained in A, be denoted as $\widetilde{A}$. Show that, in general, $\alpha(\widetilde{A}) \not\subset \widetilde{(\alpha(A))}$. What happens if α is injective? Surjective?

Sol. (1) counterexample: Let $S = \{1, 2, 3\}$, $A = \{1\}$ and $T = \{a, b\}$. Define $S \xrightarrow{\alpha} T$ by $\alpha(1) = a$, $\alpha(2) = a$, $\alpha(3) = b$. Then $\alpha(\widetilde{A}) = \{a, b\}$, $\widetilde{\{\alpha(A)\}} = \{b\}$. Moreover, in this example α is surjective.

(2) If α is injective, then $\alpha(\widetilde{A}) \subset \widetilde{(\alpha(A))}$ holds.

Proof. ¿From $A \cap \widetilde{A} = \emptyset$, we have $\alpha(\widetilde{A}) \subset \widetilde{(\alpha(A))} = \emptyset$, since α is injective. Hence $\alpha(\widetilde{A}) \subset \widetilde{(\alpha(A))}$.

But the other inclusion $\widetilde{(\alpha(A))} \subset \alpha(\widetilde{A})$ need not holds. For example, let $S = \{1, 2\}$, $A = \{1\}$ and $T = \{a, b, c\}$. Define $\alpha : S \to T$ by $\alpha(1) = a$, $\alpha(2) = b$. Then $\widetilde{(\alpha(A))} = \{b, c\}$ and $\alpha(\widetilde{A}) = \{b\}$.

(3) If α is surjective, then $\widetilde{\alpha(A)} \subseteq \alpha(\widetilde{A})$. $\square$

Proof. Let $t \in \widetilde{\alpha(A)}$. Since α is surjective, $t = \alpha(S)$ for some $s \in S$. Then $s \notin A$. Hence $t \in \alpha(\widetilde{A})$. $\square$