

3.3 Continuous Distribution

3.3.1 Uniform Distribution

The continuous uniform distribution is defined by spreading mass uniformly over an interval $[a, b]$. Its pdf given by

$$f(x|a, b) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that $\int_a^b f(x)dx = 1$ and

$$\begin{aligned} EX &= \int_a^b \frac{x}{b-a} dx = \frac{b+a}{2} \\ \text{Var}X &= \int_a^b \frac{1}{b-a} \left(x - \frac{b+a}{2}\right)^2 dx = \frac{(b-a)^2}{12}. \end{aligned}$$

3.3.2 Gamma Distribution

The pdf of gamma(α, β) distribution is defined as

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty, \quad \alpha > 0, \quad \beta > 0.$$

The parameter α is known as the shape parameter, since it most influences the peakedness of the distribution, while the parameter β is called the scale parameter, since most of its influence is on the spread of the distribution.

$\Gamma(\alpha)$ is called the gamma function, it is defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

It satisfies the following recursive formula

$$\begin{aligned} \Gamma(\alpha + 1) &= \alpha\Gamma(\alpha), \quad \alpha > 0. \\ EX &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x x^{\alpha-1} e^{-x/\beta} dx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \beta^{\alpha+1} = \alpha\beta \end{aligned}$$

In a manner analogous to the above, we can calculate EX^2 and then get

$$\text{Var}X = \alpha\beta^2.$$

The mgf of a gamma(α, β) distribution is

$$M_X(t) = \left(\frac{1}{1 - \beta t}\right)^\alpha, \quad t < \frac{1}{\beta}.$$

Example 3.3.1 (Gamma-Poisson relationship)

There is an interesting relationship between the gamma and Poisson distributions. If X is a gamma(α, β) random variable, where α is an integer, then for any x ,

$$P(X \leq x) = P(Y \geq \alpha), \tag{1}$$

where $Y \sim \text{Poisson}(x/\beta)$.

There are a number of important special cases of the gamma distribution.

Chi-squared distribution If we set $\alpha = p/2$, where p is an integer, and $\beta = 2$, then the gamma pdf becomes

$$f(x|p) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2}, \quad 0 < x < \infty,$$

which is the chi-squared pdf with p degrees of freedom.

exponential distribution If we set $\alpha = 1$, we have

$$f(x|\beta) = \frac{1}{\beta} e^{-x/\beta}, \quad 0 < x < \infty,$$

the exponential pdf with scale parameter β . The exponential distribution can be used to model lifetimes, analogous to the use of the geometric distribution in the discrete case. In fact, the exponential distribution shares the “memoryless” property of the geometric.

$$\begin{aligned} P(X > s|X > t) &= \frac{P(X > s, X > t)}{P(X > t)} = \frac{P(X > s)}{P(X > t)} \\ &= \frac{\int_s^\infty \frac{1}{\beta} e^{-x/\beta} dx}{\int_t^\infty \frac{1}{\beta} e^{-x/\beta} dx} = \frac{e^{-s/\beta}}{e^{-t/\beta}} \\ &= e^{-(s-t)/\beta} = P(X > s - t). \end{aligned}$$

Weibull distribution If $X \sim \text{exponential}(\beta)$, then $Y = X^{1/\gamma}$ has a Weibull(γ, β) distribution,

$$f_Y(y|\gamma, \beta) = \frac{\gamma}{\beta} y^{\gamma-1} e^{-y^\gamma/\beta}, \quad 0 < y < \infty, \quad \gamma > 0, \quad \beta > 0.$$

The Weibull distribution plays an extremely important role in the analysis of failure time data.