3.2.5 Negative Binomial Distribution

In a sequence of independent Bernoulli(p) trials, let the random variable X denote the trial at which the r^{th} success occurs, where r is a fixed integer. Then

$$P(X = x|r, p) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \ldots, \tag{1}$$

and we say that X has a negative binomial(r, p) distribution.

The negative binomial distribution is sometimes defined in terms of the random variable Y =number of failures before rth success. This formulation is statistically equivalent to the one given above in terms of X =trial at which the rth success occurs, since $Y = X - r$. The alternative form of the negative binomial distribution is

$$P(Y = y) = \binom{r+y-1}{y} p^r (1-p)^y, \quad y = 0, 1, \ldots.$$

The negative binomial distribution gets its name from the relationship

$$\binom{r+y-1}{y} = (-1)^y \binom{-r}{y} = (-1)^y \frac{(-r)(-r-1)\cdots(-r-y+1)}{(y)(y-1)\cdots(2)(1)}, \tag{2}$$

which is the defining equation for binomial coefficient with negative integers. Along with (2), we have

$$\sum_y P\big(Y = y\big) = 1$$

from the negative binomial expansition which states that

$$\begin{aligned} (1+t)^{-r} &= \sum_k \binom{-r}{k} t^k \\ &= \sum_k (-1)^k \binom{r+k-1}{k} t^k \end{aligned}$$

$$
\begin{aligned}
EY &= \sum_{y=0}^{\infty} y \binom{r+y-1}{y} p^r (1-p)^y \\
&= \sum_{y=1}^{\infty} \frac{(r+y-1)!}{(y-1)!(r-1)!} p^r (1-p)^y \\
&= \sum_{y=1}^{\infty} \frac{r(1-p)}{p} \binom{r+y-1}{y-1} p^{r+1} (1-p)^{y-1} \\
&= \frac{r(1-p)}{p} \sum_{z=0}^{\infty} \binom{r+1+z-1}{z} p^{r+1} (1-p)^z \\
&= r \frac{1-p}{p}.
\end{aligned}
$$

A similar calculation will show

$$
\mathrm{Var} Y = \frac{r(1-p)}{p^2}.
$$

Example 3.2.6 (Inverse Binomial Sampling

A technique known as an inverse binomial sampling is useful in sampling biological populations. If the proportion of individuals possessing a certain characteristic is p and we sample until we see r such individuals, then the number of individuals sampled is a negative bnomial rndom variable.

0.1 Geometric distribution

The geometric distribution is the simplest of the waiting time distributions and is a special case of the negative binomial distribution. Let $r = 1$ in (1) we have

$$
P(X = x|p) = p(1-p)^{x-1}, \quad x = 1, 2, \ldots,
$$

which defines the pmf of a geometric random variable X with success probability p.

X can be interpreted as the trial at which the first success occurs, so we are "waiting for a success". The mean and variance of X can be calculated by using the negative binomial formulas and by writing $X = Y + 1$ to obtain

$$
EX = EY + 1 = \frac{1}{P} \quad \text{and} \quad \mathrm{Var} X = \frac{1-p}{p^2}.
$$

The geometric distribution has an interesting property, known as the "memoryless" property. For integers $s > t$, it is the case that

$$P(X > s|X > t) = P(X > s - t), \tag{3}$$

that is, the geometric distribution "forgets" what has occurred. The probability of getting an additional $s - t$ failures, having already observed t failures, is the same as the probability of observing $s - t$ failures at the start of the sequence.

To establish (3), we first note that for any integer n,

$$P(X > n) = P(\text{no success in } n \text{ trials}) = (1 - p)^n,$$

and hence,

$$\begin{aligned} P(X > s|X > t) &= \frac{P(X > s \text{ and } X > t)}{P(X > t)} = \frac{P(X > s)}{P(X > t)} \\ &= (1 - p)^{s-t} = P(X > s - t). \end{aligned}$$