3.2.4 Poisson Distribution

Definition Let X be **the number of events** per **basic unit**: For example,

- Number of rain drops in one minute.
- Number of cars passing by you for an hour.
- Number of chocolate particles in one ChoCoChip cookie.
- Number of typos in one page.
- Number of defaults in one cm^2 area.
- Number of visitors of a certain web site between 10:00-11:00pm.
- Number of characters in Page 256 of the text book.

The Poisson distribution has a single parameter λ, sometimes called the intensity parameter. A random variable X, taking values in the nonnegative integers, has a Poisson(λ) distribution if

$$P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots$$

The mean of X is

$$\begin{aligned} EX &= \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!} \\ &= \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} \\ &= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^{y}}{y!} = \lambda \end{aligned}$$

A similar calculation will show that

$$\mathrm{Var}X = \lambda.$$

The mgf is

$$M_X(t) = e^{\lambda(e^t-1)}.$$

Example 3.2.4 (Waiting time)

As an example of a waiting-for-occurrence application, consider a telephone operator who, on the average, handles five calls every 3 minutes. What is the probability that there will be no calls in the next minute? At least two calls?

If we let X =number of calls in a minute, then X has a Poisson distribution with $EX = \lambda = 5/3$. So

$$P(\text{no calls in the next minute}) = P(X = 0)$$
$$= \frac{e^{-5/3}(\frac{5}{3})^0}{0!} = e^{-5/3} = 0.189$$

$$P(\text{at least two calls in the next minute}) = P(X \geq 2)$$
$$= 1 - P(X = 0) - P(X = 1) = 1 - .189 - \frac{e^{-5/3}(5/3)^1}{1!} = 0.496.$$

Example 3.2.5 (Poisson approximation)

A typesetter, on the average, makes one error in every 500 words typeset. A typical page contains 300 words. What is the probability that there will be no more than two errors in five pages?

If we assume that setting a word is a Bernoulli trial with success probability $p = \frac{1}{500}$, then X =number of errors in five pages (1500 words) is binomial(1500, $\frac{1}{500}$). Thus,

$$P(\text{no more than two errors}) = P(X \leq 2)$$
$$= \sum_{x=0}^{2} \binom{1500}{x} (\frac{1}{500})^x (\frac{499}{500})^{1500-x}$$
$$= .4230.$$

If we use the Poisson approximation with $\lambda = 1500/500 = 3$, we have

$$P(X \leq 2) \approx e^{-3}(1 + 3 + \frac{3^2}{2}) = 0.4232.$$