Lecture 1: Set Theory

1 Set Theory

One of the main objectives of a statistician is to draw conclusions about a population of objects by conducting an experiment. The fist step in this endeavor is to identify the possible outcomes or, in statistical terminology, the sample space.

Definition 1.1 *The set, S, of all possible outcomes of a particular experiment is called the sample space for the experiment.*

If the experiment consists of tossing a coin, the sample space contains two outcomes, heads and tails; thus, $S = \{H, T\}$.

Consider an experiment where the observation is reaction time to a certain stimulus. Here, the sample space would consists of all positive numbers, that is, $S = (0, \infty)$.

The sample space can be classified into two type: countable and uncountable. If the elements of a sample space can be put into 1–1 correspondence with a subset of integers, the sample space is countable. Otherwise, it is uncountable.

Definition 1.2 *An event is any collection of possible outcomes of an experiment, that is, any subset of S (including S itself).*

Let A be an event, a subset of S. We say the event A occurs if the outcome of the experiment is in the set A.

We first define two relationships of sets, which allows us to order and equate sets:

$$A \subset B \Leftrightarrow x \in A \Rightarrow x \in B \qquad \text{(containment)}$$

$$A = B \Leftrightarrow A \subset B \quad \text{and} \quad B \subset A. \qquad \text{(equality)}$$

Given any two events (or sets) A and B, we have the following elementary set operations:

Union: The union of A and B, written $A \cup B$, is the set of elements that belong to either A or B or both:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Intersection: The intersection of A and B, written $A \cap B$, is the set of elements that belong to both A and B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Complementation: The complement of A, written A^c, is the set of all elements that are not in A:

$$A^c = \{x : x \notin A\}.$$

Example 1.1 *(Event operations) Consider the experiment of selecting a card at random from a standard deck and noting its suit: clubs (C), diamond (D), hearts (H), or spades (S). The sample space is*

$$S = \{C, D, H, S\},$$

and some possible events are

$$A = \{C, D\}, \quad \textit{and} \quad B = \{D, H, S\}.$$

From these events we can form

$$A \cup B = \{C, D, H, S\}, \quad A \cap B = \{D\}, \quad \textit{and} \quad A^c = \{H, S\}.$$

Furthermore, notice that $A \cup B = S$ and $(A \cup B)^c = \emptyset$, where denotes the empty set (the set consisting of no elements).

Theorem 1.1 *For any three events, A, B, and C, defined on a sample space S,*

a. Commutativity. $A \cup B = B \cup A$, $A \cap B = B \cap A$.

b. Associativity. $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

c. Distributive Laws. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

d. DeMorgan's Laws. $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

The operations of union and interaction can be extended to infinite collections of sets as well. If $A_1, A_2, A_3, \ldots$ is a collection of sets, all defined on a sample space S, then

$$\cup_{i=1}^{\infty} A_i = \{x \in S : x \in A_i \quad \text{for some } i\}.$$

$$\cap_{i=1}^{\infty} A_i = \{x \in S : x \in A_i \quad \text{for all } i\}.$$

For example, let $S = (0, 1]$ and define $A_i = [(1/i), 1]$. Then

$$\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} [(1/i), 1] = (0, 1]$$

$$\cap_{i=1}^{\infty} A_i = \cap_{i=1}^{\infty} [(1/i), 1] = \{1\}.$$

It is also possible to define unions and intersections over uncountable collections of sets. If Γ is an index set (a set of elements to be used as indices), then

$$\cup_{a \in \Gamma} A_a = \{x \in S : x \in A_a \ \text{ for some } a\},$$

$$\cap_{a \in \Gamma} A_a = \{x \in S : x \in A_a \ \text{ for all } a\}.$$

Definition 1.3 *Two events A and B are disjoint (or mutually exclusive) if $A \cap B = \emptyset$. The events $A_1, A_2, \ldots$ are pairwise disjoint (or mutually exclusive) if $A_i \cap A_j = \emptyset$ for all $i \neq j$.*

Definition 1.4 *If $A_1, A_2, \ldots$ are pairwise disjoint and $\cup_{i=1}^{\infty} A_i = S$, then the collection $A_1, A_2, \ldots$ forms a partition of S.*