Advanced Statistical Inference I

Homework 2: Transformations and Expectations

Due Date: October 19th

1. (transformation)
   - **(a)** Exercise 2.2(c).
   - **(b)** Exercise 2.3.
   - **(c)** Exercise 2.23(a).
2. (Simulation)
   - **(a)** Exercise 2.8(b).
   - **(b)** Exercise 2.9.
   - **(c)** Exercise 2.10.
3. (Median)
   - **(a)** Exercise 2.18.
   - **(b)** When the median of a random variable X (or its distribution) is any value m such that $P(X \geq m) \geq 1/2$ and $P(X \leq m) \geq 1/2$, show that the set of medians is a closed interval $[m_0, m_1]$.
4. (Data summary) For any set of numbers $x_1, \ldots, x_n$ and a monotone function $h(\cdot)$, show that the value of a that minimizes $\sum_{i=1}^n [h(x_i) - h(a)]^2$ is given by $a = h^{-1}(\sum_{i=1}^n h(x_i)/n)$. Find functions h that will yield the arithmetic, geometric, and harmonic means as minimizes.
   Recall that the geometric mean of non-negative numbers is $\left(\prod_{i=1}^n x_i\right)^{1/n}$ and the harmonic mean is $\left[n^{-1} \sum_{i=1}^n (1/x_i)\right]^{-1}$.
5. Let X be an absolutely continuous random variable with cdf F and having variance σ^2.
   - **(a)** Show that
   $$\sigma^2 = \frac{1}{2} \int \int_{-\infty < x < y < \infty} F(x)[1 - F(y)] dx dy.$$
   (Hint: Try integration by parts.)
   - **(b)** Replace F by the sample distribution function F_n in (a). Is it closely related to the commonly used sample variance?
6. (Moment generating function)
   - **(a)** Exercise 2.31.
   - **(b)** Exercise 2.32.
   - **(c)** Exercise 2.38.
7. (Moments and tail probability)
   - **(a)** Exercise 2.14.
   - **(b)** Let X be a random variable and $a > 0$. Show that $E|X|^a < \infty$ if and only if $\sum_{n=1}^\infty n^{a-1} P(|X| \geq n) < \infty$.