Large Sample Theory

Homework 4: Methods of Estimation, Asymptotic Distribution, Probability and Conditioning

Due Date: December 1st

1. The Weibull distribution (after the Swedish physicist Waloddi Weibull, who proposed the distribution in 1939 for the breaking strength of materials), has density function

$$f(x) = \lambda x^{\lambda-1} \exp\left(-x^{\lambda}\right) \quad \text{for } x, \lambda > 0.$$

   [As an aside, note that the Weibull arises by assuming $y = x^{\lambda}$Mfollows an exponential distribution].

   a. What is the resulting likelihood function $\ell(\lambda|x_1, \ldots, x_n)$, for λ?
   b. What is the resulting log-likelihood function?
   c. What is the score function?
   d. What is the second derivative of the log-likelihood function?
   e. Suppose 5 values, $0.10, 0.25, 0.5, 1$, and 2 are observed. Plot the resulting log-likelhood function
   f. What is the approximate sample variance?
   g. What is an approximate 95% confidence interval for λ?

2. Let X be $N(0, \theta)$, $0 < \theta < \infty$.

   a. Find the Fisher information $I(\theta)$.
   b. If $X_1, X_2, \ldots, X_n$ is a random sample from this distribution, show that the MLE of θ is an efficient estimator of θ.

3. For Type II censoring, the data consist of the rth smallest lifetimes $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ out of a random sample of n lifetimes $X_1, \ldots, X_n$ from the assumed life distribution. Assuming $X_1, \ldots, X_n$ are i.i.d. and have a continuous distribution with p.d.f. $f(x)$ and survival function $S(x)$.

   a. Show that the joint p.d.f. of $X_{(1)}, X_{(2)}, \cdots X_{(r)}$ is

$$L_{II,1} = \frac{n!}{(n-r)!} \left[\prod_{i=1}^{r} f(x_{(i)})\right] [S(x_{(r)}]^{n-r}.$$

   b. Suppose that X_i is an exponentially distributed random variable with mean θ. Derive the MLE of θ, $\hat{\theta}$, and state the condition on r to guarantee consistency of $\hat{\theta}$.
   c. Use EM algorithm to derive the MLE of θ.

4. The normally distributed random variables $X_1, \ldots, X_n$ are said to be serially correlated or to follow an autoregressive model if we can write

$$X_i = \theta X_{i-1} + \epsilon_i, \quad i = 1, \ldots, n,$$

   where $X_0 = 0$ and $\epsilon_1, \ldots, \epsilon_n$ are independent $N(0, \sigma^2)$ random variables.

   a. Show that the density of $(X_1, \ldots, X_n)$ is

$$\frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{\sum_{i=1}^{n}(x_i - \theta x_{i-1})^2}{2\sigma^2}\right\}$$

   for $-\infty < x_i < \infty$, $i = 1, \ldots, n$, $x_0 = 0$.

b. Derive MLE of θ and σ^2. Give a condition on θ so that they are consistent estimates.

5. Let Y_i denote the response of a subject at time i, $i = 1, \dots, n$. Suppose that Y_i satisfies the following model

$$Y_i = \theta + \epsilon_i, \quad i = 1, \dots, n$$

where ϵ_i can be written as $\epsilon_i = ce_{i-1} + e_i$ for a given constant c satisfying $0 \le c \le 1$, and the e_i are independent and identically distributed with mean zero and variance σ^2, $i = 1, \dots, n$; $\epsilon_0 = 0$. Let

$$\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i, \quad \hat{\theta} = \sum_{j=1}^{n} a_j Y_j$$

where

$$a_j = \sum_{i=0}^{n-j} (-c)^j \left(\frac{1 - (-c)^{j+1}}{1+c} \right) / \sum_{i=1}^{n} \left(\frac{1 - (-c)^i}{1+c} \right)^2 .$$

a. Show that if $e_i \sim N(0, \sigma^2)$, then $\hat{\theta}$ is the MLE of θ.

b. Show that $\bar{Y}$ and $\hat{\theta}$ are unbiased.

c. Show that $Var(\bar{Y}) \ge Var(\hat{\theta})$.

d. Show that $\bar{Y}$ and $\hat{\theta}$ are consistent estimates of θ.

6. Suppose that $X_1, \dots, X_n$ are independent and identically distributed according to a location family with cdf $F(x - \theta)$, with F known and with $0 < F(x) < 1$ for all x, but that it is only observed whether each X_i falls below a, between a and b, or above b where $a < b$ are two given constants.

a. Describe the joint distribution of the observed three outcomes.

b. Let V denote the number of observations less than a. Describe the asymptotic distribution of $\sqrt{n}(V/n - p_1)$ where $p_1 = F(a - \theta)$.

c. Show that $\tilde{V}_n = a - F^{-1}(V/n)$ is a consistent estimate of θ. Derive the asymptotic distribution of $\sqrt{n}(\tilde{V}_n - \theta)$

7. Let $X_1, \dots, X_n$ be iid with distribution P_θ depending on a real-valued parameter θ, and suppose that $E_\theta(X) = g(\theta)$ and $Var_\theta(X) = \tau(\theta) < \infty$, where g is continuously differentiable function with derivative $g'(\theta) > 0$ for all θ. Denote the estimator obtained by the method of moments by $\hat{\theta}$. (i.e., $\hat{\theta}$ is the solution of the equation $g(\theta) = \bar{X}$.)

a. Show that $\hat{\theta}$ is consistent.

b. Derive its asymptotic distribution.

8. Suppose that v_i and u_i, $1 \le i \le n$, are associated with a linear relationship $v_i = a + bu_i$. Due to data collection error, we can only observe (x_i, y_i) where $y_i = v_i + \delta_i$ and $x_i = u_i + \epsilon_i$. It is known that $E(\delta_i) = E(\epsilon_i) = 0$ and δ_i and ϵ_i are to be independent. Note that $y_i = a + bx_i + (\delta_i - b\epsilon_i)$ and $E(\delta_i - b\epsilon_i) = 0$.

a. When $Var(\epsilon_i) = Var(\delta_i) = \sigma^2$, show that the least squares estimate of b (based on (x_i, y_i)) is not consistent when $n^{-1} \sum_{i=1}^{n} (u_i - \bar{u})^2$ goes to a nonzero constant c.

b. Propose a consistent estimate of b when $Var(\delta_i) = 2Var(\epsilon_i)$.

9. Let $X_1, \ldots, X_n$ be iid according to the normal distribution $N(\theta, 1)$. Consider the sequence of estimators

$$\delta_n = \begin{cases} \bar{X} & \text{if } |\bar{X}| \geq n^{-1/4} \\ a\bar{X} & \text{if } |\bar{X}| < n^{-1/4} \end{cases}$$

Find the asymptotic distribution of $\sqrt{n}(\delta_n - \theta)$.
Hint: You may need to derive your answer for $\theta = 0$ and $\theta \neq 0$ separately.

10. Show the following properties of the multivariate normal distribution $N_k(\mu, \Sigma)$ where $\mu \in R^k$ and Σ is a positive definite $k \times k$ matrix. Note that, if $\mathbf{X} \sim N_k(\mu, \Sigma)$, its pdf is

$$f(\mathbf{x}) = (2\pi)^{-k/2}[Det(\Sigma)]^{-1/2} \exp(-(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)).$$

(a) The mgf of $N_k(\mu, \Sigma)$ is $\exp(\mu^T \mathbf{t} + \mathbf{t}^T \Sigma \mathbf{t}/2)$.
Fact: The mgf of $\mathbf{X}$ is defined as $E \exp(\mathbf{X}^T \mathbf{t})$.
(b) Let $\mathbf{X}$ be a random k-vector having the $N_k(\mu, \Sigma)$ distribution and $\mathbf{Y} = A\mathbf{X} + c$, where A is a $k \times \ell$ matrix of rank $\ell \leq k$ and $c \in R^\ell$. Then $\mathbf{Y}$ has the $N_\ell(A\mu + c, A^T \Sigma A)$ Distribution.
Fact: If $\mathbf{X}$ and $\mathbf{Y}$ are random k-vectors and their mgf are identical for all $\mathbf{t} \in N_\epsilon = \{\mathbf{t} \in R^k : \|t\| \leq \epsilon\}$, then the distribution of $\mathbf{X}$ is identical to that of $\mathbf{Y}$.
(c) A random k-vector $\mathbf{X}$ has a k-dimensional normal distribution if and only if for any $c \in R^k$, $\mathbf{X}^T \mathbf{c}$ has a univariate normal distribution.
(d) Let $\mathbf{X}$ be a random k-vector having the $N_k(\mu, \Sigma)$ distribution. Let A be a $k \times \ell$ matrix and B be a $k \times m$ matric. Then $\mathbf{X}A$ and $\mathbf{X}B$ are independent if and only if they are uncorrelated.
(e) Let $(\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ be a random k-vector having the $N_k(\mu, \Sigma)$ distribution with

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\mathbf{X}_1$ is a random ℓ-vector and Σ_{11} is an $\ell \times \ell$ matrix. Then the conditional pdf of $\mathbf{X}_2$ given $\mathbf{X}_1$ is

$$N_{k-\ell}(\mu_2 + (\mathbf{x}_1 - \mu_1)\Sigma_{11}^{-1}\Sigma_{12}, \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}),$$

where $\mu_i = E(\mathbf{X}_i)$, $i = 1, 2$.
Hint: Consider $\mathbf{X}_2 - \mu_2 - (\mathbf{X}_1 - \mu_1)\Sigma_{11}^{-1}\Sigma_{12}$ and $\mathbf{X}_1 - \mu_1$.)

11. Suppose X_1, X_2, and X_3 are multivariate normally distributed with means 1 $\mu_1 = 1$, $\mu_2 = 0$, $\mu_3 = -2$ and covariance structure

$$\sigma^2(X_1) = 3, \ \ \sigma^2(X_2) = 4, \ \ \sigma^2(X_3) = 6, \ \ \sigma(X_1, X_2) = 1, \ \ \sigma(X_1, X_3) = -1, \ \ \sigma(X_2, X_3) = 2.$$

   a. What is the distribution of (X_1, X_2) given X_3?
   b. What is the regression of X_1 on X_2 and X_3?
   c. What is the conditional variance of X_1 given X_2 and X_3?