GARCH AND STOCHASTIC VOLATILITY OPTION PRICING

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   - Diffusion limit of the GARCH option pricing model
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1. Black-Scholes Model (1973)

- **Asset price process**

$$d\ln(S_t) = (r + \lambda\sigma - \frac{\sigma^2}{2})dt + \sigma dW_t$$

- **Risk-neutralized asset price process**

$$d\ln(S_t) = (r - \frac{\sigma^2}{2})dt + \sigma dW_t^*$$

- **Pricing formula**

For a European <u>call</u> option payoff at time *T*,

$$Max(S_T - K, 0)$$

its time-0 value is by the closed-form solution

$$\begin{aligned} &C(S_0; K, T, r, \sigma) \\ &= S_0 N(d) - Ke^{-rT} N(d - \sigma\sqrt{T}) \end{aligned}$$

where

$$d = \frac{\ln\frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

- **Implied Volatility**

Find $\sigma^*(K_i, T_j)$ to solve

$$C^{mkt}(K_i, T_j) = C(S_0; K_i, T_j, r, \sigma^*(K_i, T_j))$$

- **Implied volatility vs. historical volatility**

  If the Black-Scholes model works well, the implied volatility should be roughly the same as the historical volatility. (March 11, 1998, South China Morning Post; Historical volatility (250 days) equals 46%.)

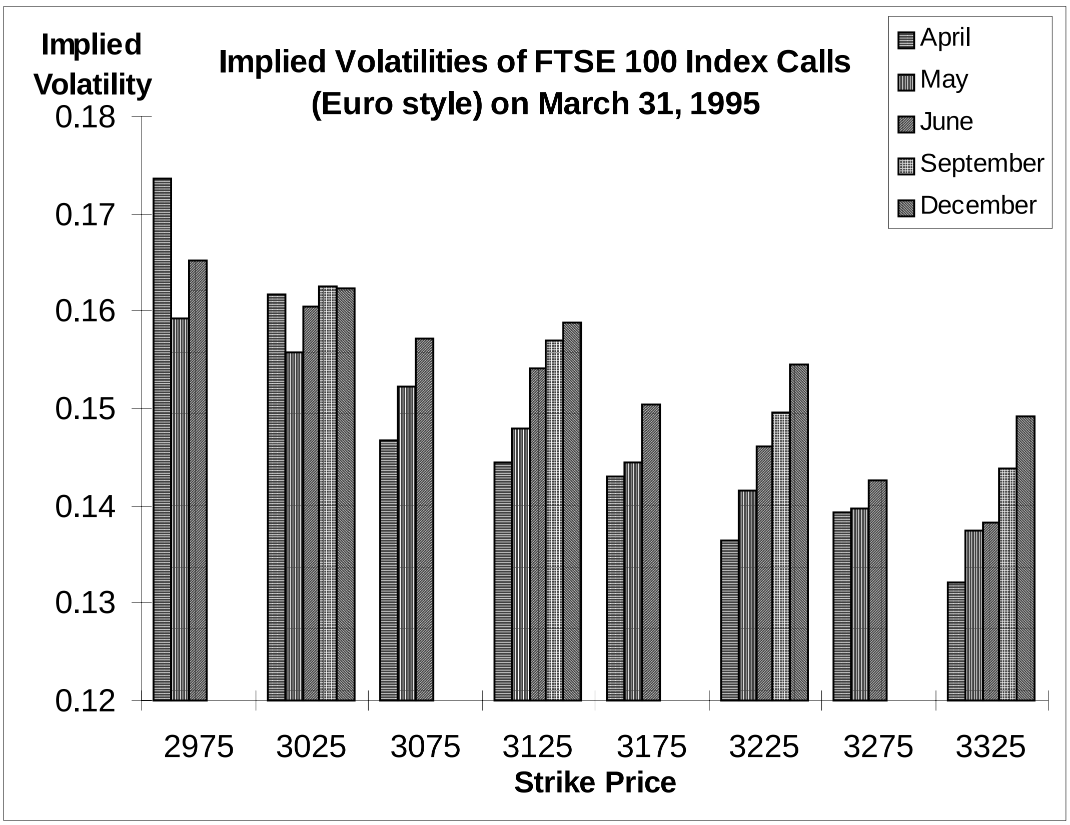
HANG SENG INDEX OPTIONS

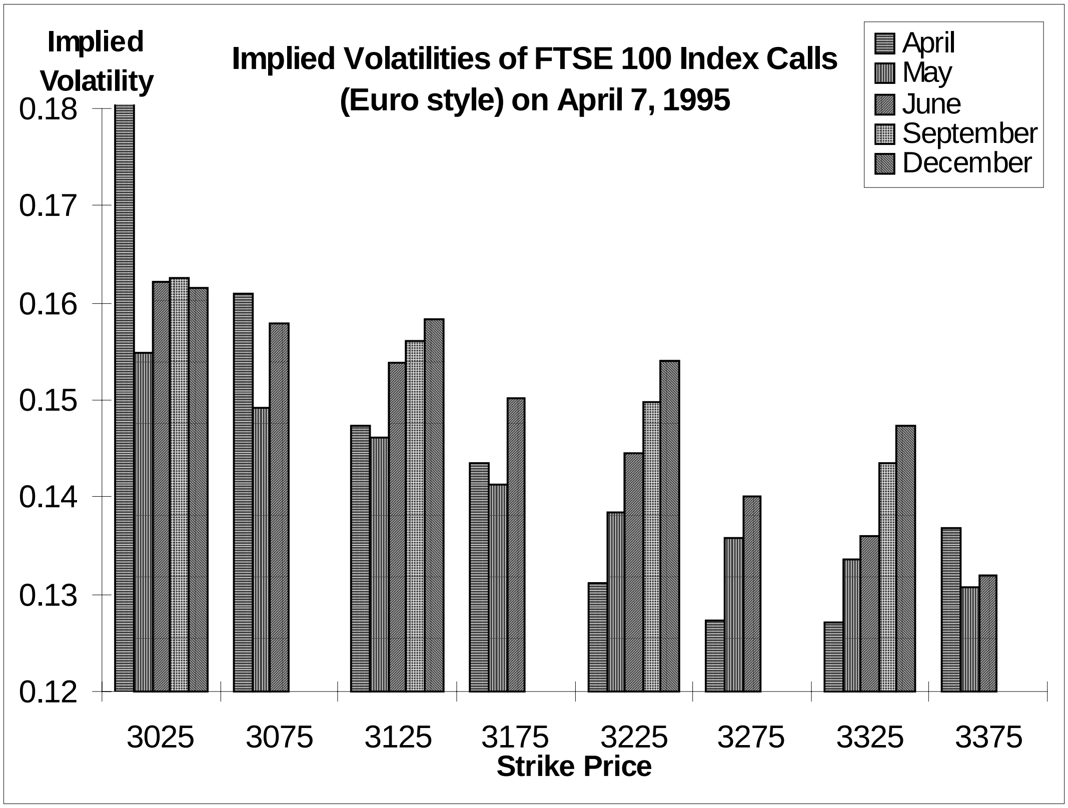
Months	Strike		Close	change	Volit %	Est. Vol	Open int.	Months	Strike		Close	change	Volit %	Est. Vol	Open int.
Mar/98	7800	C	3409	+234	65	0	0	Mar/98	7800	P	9	-6	75	0	3
Mar	7900	C	3310	+233	65	0	0	Mar	7900	P	10	-7	74	0	0
Mar	8000	C	3212	+233	65	0	0	Mar	8000	P	12	-7	73	0	132
Mar	8200	C	3015	+231	65	0	0	Mar	8200	P	15	-9	71	0	112
Mar	8400	C	2819	+228	65	0	0	Mar	8400	P	19	-12	69	0	339
Mar	8600	C	2625	+226	65	0	0	Mar	8600	P	25	-14	68	0	38
Mar	8800	C	2432	+223	65	0	0	Mar	8800	P	32	-17	66	0	1208
Mar	9000	C	2240	+219	65	0	178	Mar	9000	P	40	-21	64	7	991
Mar	9200	C	2051	+214	61	0	0	Mar	9200	P	51	-26	62	7	330
Mar	9400	C	1864	+209	61	0	192	Mar	9400	P	64	-31	60	1	581
Mar	9600	C	1681	+203	57	0	100	Mar	9600	P	81	-37	59	2	686
Mar	9800	C	1501	+195	57	0	250	Mar	9800	P	101	-45	57	20	496
Mar	10000	C	1326	+186	55	0	788	Mar	10000	P	126	-54	55	284	1666
Mar	10200	C	1157	+176	53	0	379	Mar	10200	P	157	-64	53	149	911
Mar	10400	C	994	+164	51	0	992	Mar	10400	P	194	-76	51	56	930
Mar	10600	C	840	+151	49	0	834	Mar	10600	P	240	-89	50	43	1279
Mar	10800	C	695	+136	48	0	1011	Mar	10800	P	295	-104	48	29	590
Mar	11000	C	561	+121	46	26	717	Mar	11000	P	361	-119	46	42	1084
Mar	11200	C	440	+92	44	61	669	Mar	11200	P	440	-148	44	41	450
Mar	11400	C	345	+76	43	23	1072	Mar	11400	P	545	-164	43	11	169
Mar	11600	C	265	+61	43	24	364	Mar	11600	P	665	-179	43	0	9
Mar	11800	C	198	+48	42	21	292	Mar	11800	P	798	-192	43	0	5
Mar	12000	C	144	+36	42	162	1230	Mar	12000	P	944	-204	42	0	168
Mar	12200	C	101	+26	41	4	78	Mar	12200	P	1101	-214	41	0	100
Mar	12400	C	69	+18	40	17	1894	Mar	12400	P	1269	-222	41	0	200
Mar	12600	C	46	+13	40	60	810	Mar	12600	P	1446	-227	41	0	100
Mar	12800	C	29	+8	39	3	1660	Mar	12800	P	1629	-232	41	0	475
Mar	13000	C	18	+5	39	2	2071	Mar	13000	P	1818	-235	41	0	128
Mar	13200	C	10	+3	38	0	999	Mar	13200	P	2010	-237	33	0	0
Mar	13400	C	6	+2	38	0	4	Mar	13400	P	2206	-238	33	0	400
Apr/98	9800	C	1636	+173	48	0	0	Apr/98	9800	P	226	-67	48	3	0
Apr	10000	C	1479	+164	47	0	0	Apr	10000	P	269	-76	47	225	256
Apr	10200	C	1328	+155	46	0	0	Apr	10200	P	318	-85	47	0	201
Apr	10400	C	1183	+144	45	0	3	Apr	10400	P	373	-96	46	0	150
Apr	10600	C	1047	+134	45	0	5	Apr	10600	P	437	-106	45	0	239
Apr	10800	C	918	+123	44	100	102	Apr	10800	P	508	-117	44	0	2
Apr	11000	C	797	+112	43	0	351	Apr	11000	P	587	-128	43	16	10
Apr	11200	C	685	+91	42	0	200	Apr	11200	P	675	-149	42	0	0
Apr	11400	C	592	+80	42	0	200	Apr	11400	P	782	-160	42	0	0
Apr	11600	C	508	+70	42	0	6	Apr	11600	P	898	-170	42	0	6
Apr	11800	C	433	+61	42	0	1200	Apr	11800	P	1023	-179	42	0	0
Apr	12000	C	366	+52	42	0	1	Apr	12000	P	1156	-188	42	0	0
Apr	12200	C	307	+43	41	0	0	Apr	12200	P	1297	-197	42	0	0
Apr	12400	C	256	+36	41	3	400	Apr	12400	P	1446	-204	42	0	0
Apr	12600	C	212	+30	41	0	301	Apr	12600	P	1602	-210	41	0	0
Apr	12800	C	174	+25	41	1	300	Apr	12800	P	1764	-215	41	0	0
May/98	9600	C	1943	+170	48	0	0	May/98	9600	P	313	-60	48	0	0
May	9800	C	1790	+163	47	0	0	May	9800	P	360	-67	48	0	0
May	10000	C	1641	+156	47	0	0	May	10000	P	411	-74	47	0	0
May	10200	C	1498	+149	46	0	0	May	10200	P	468	-81	46	0	0
May	10400	C	1361	+141	45	0	0	May	10400	P	531	-89	45	0	0
May	10600	C	1230	+133	44	0	0	May	10600	P	600	-97	45	0	0
May	10800	C	1105	+125	44	0	0	May	10800	P	675	-105	44	0	0
May	11000	C	987	+117	43	0	0	May	11000	P	757	-113	43	0	0
May	11200	C	875	+96	42	0	0	May	11200	P	845	-134	43	0	0
May	11400	C	782	+88	42	0	0	May	11400	P	952	-142	42	0	0
May	11600	C	696	+80	42	0	25	May	11600	P	1066	-150	42	0	0
May	11800	C	617	+72	42	0	0	May	11800	P	1187	-158	42	0	0
May	12000	C	545	+65	42	50	0	May	12000	P	1315	-165	42	0	0
May	12200	C	479	+58	42	0	0	May	12200	P	1449	-172	42	0	0
May	12400	C	420	+52	41	0	0	May	12400	P	1590	-178	42	0	0
May	12600	C	366	+45	41	0	0	May	12600	P	1736	-185	42	0	0
May	12800	C	318	+40	41	0	0	May	12800	P	1888	-190	42	0	0
Jun/98	10000	C	1809	+173	47	0	160	Jun/98	10000	P	559	-57	47	0	525
Jun	10200	C	1671	+167	46	0	300	Jun	10200	P	621	-63	47	0	470
Jun	10400	C	1539	+162	45	0	1840	Jun	10400	P	689	-68	46	0	200
Jun	10600	C	1411	+155	45	0	275	Jun	10600	P	761	-75	45	0	75
Jun	10800	C	1289	+149	44	0	3476	Jun	10800	P	839	-81	45	0	77
Jun	11000	C	1172	+142	43	0	929	Jun	11000	P	922	-88	44	0	268
Jun	11200	C	1060	+121	43	0	517	Jun	11200	P	1010	-109	43	0	20
Jun	11400	C	969	+115	43	0	150	Jun	11400	P	1119	-115	43	0	0
Jun	11600	C	883	+107	42	0	200	Jun	11600	P	1233	-123	43	0	1
Jun	11800	C	803	+101	42	0	1	Jun	11800	P	1353	-129	43	0	0
Jun	12000	C	728	+93	42	100	165	Jun	12000	P	1478	-137	43	0	10
Jun	12200	C	659	+87	42	0	278	Jun	12200	P	1609	-143	43	0	0
Jun	12400	C	596	+81	42	0	150	Jun	12400	P	1746	-149	43	0	0
Jun	12600	C	537	+75	42	0	90	Jun	12600	P	1887	-155	43	0	1
Jun	12800	C	483	+69	42	0	0	Jun	12800	P	2033	-161	43	0	0
Jun	13000	C	433	+63	42	200	500	Jun	13000	P	2183	-167	43	0	21
Sep/98	10000	C	2161	+172	43	0	0	Sep/98	10000	P	761	-68	44	0	275
Sep	10200	C	2037	+166	43	0	0	Sep	10200	P	837	-74	43	0	0
Sep	10400	C	1918	+161	43	0	0	Sep	10400	P	918	-79	43	0	0
Sep	10600	C	1804	+157	43	0	0	Sep	10600	P	1004	-83	43	0	0
Sep	10800	C	1694	+151	42	0	50	Sep	10800	P	1094	-89	43	0	0
Sep	11000	C	1588	+146	42	0	200	Sep	11000	P	1188	-94	43	0	0
Sep	11200	C	1487	+140	42	0	200	Sep	11200	P	1287	-100	43	0	100
Sep	11400	C	1390	+131	42	0	100	Sep	11400	P	1390	-109	42	0	100
Sep	11600	C	1301	+126	42	0	0	Sep	11600	P	1501	-114	42	0	0
Sep	11800	C	1217	+121	42	0	0	Sep	11800	P	1617	-119	42	0	0
Sep	12000	C	1136	+115	42	0	65	Sep	12000	P	1736	-125	42	0	0
Sep	12200	C	1060	+110	42	0	0	Sep	12200	P	1860	-130	42	0	0
Sep	12400	C	988	+105	41	0	150	Sep	12400	P	1988	-135	42	0	0

- **Volatility Smile**

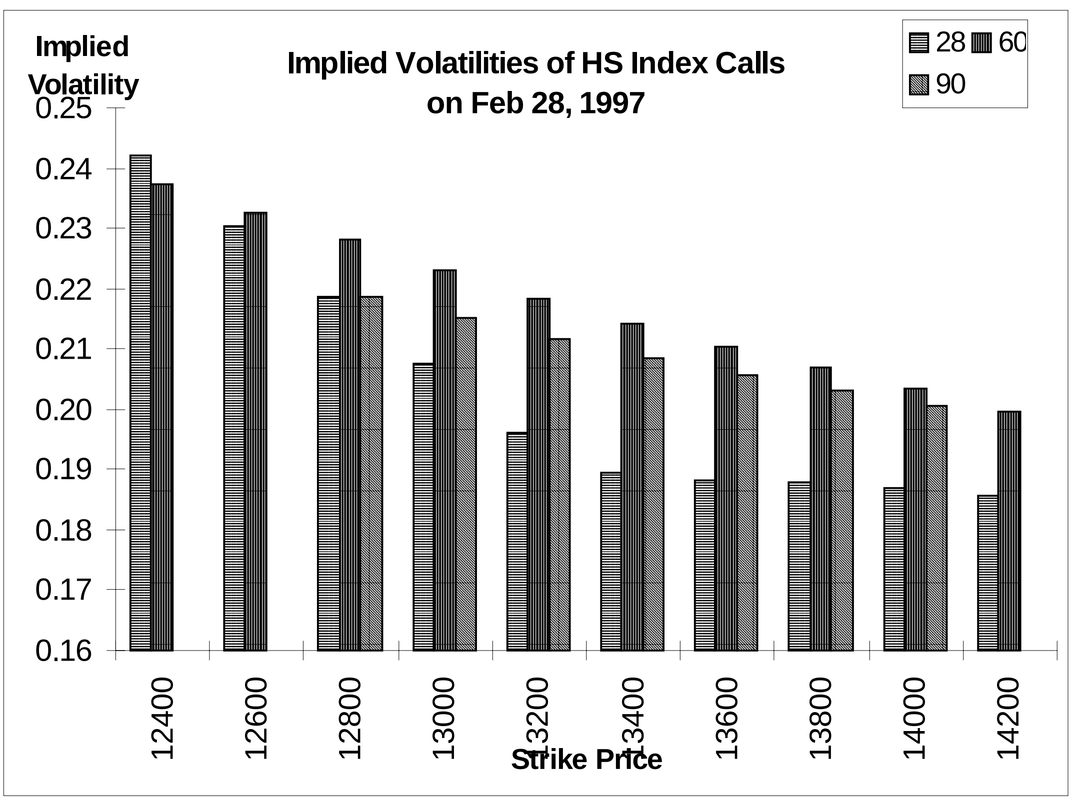
If the Black-Scholes model works well, implied volatility $\sigma^*(K_i, T_j)$ should be independent of K_i and T_j. In reality, $\sigma^*(K_i, T_j)$ varies systematically with K_i and T_j.

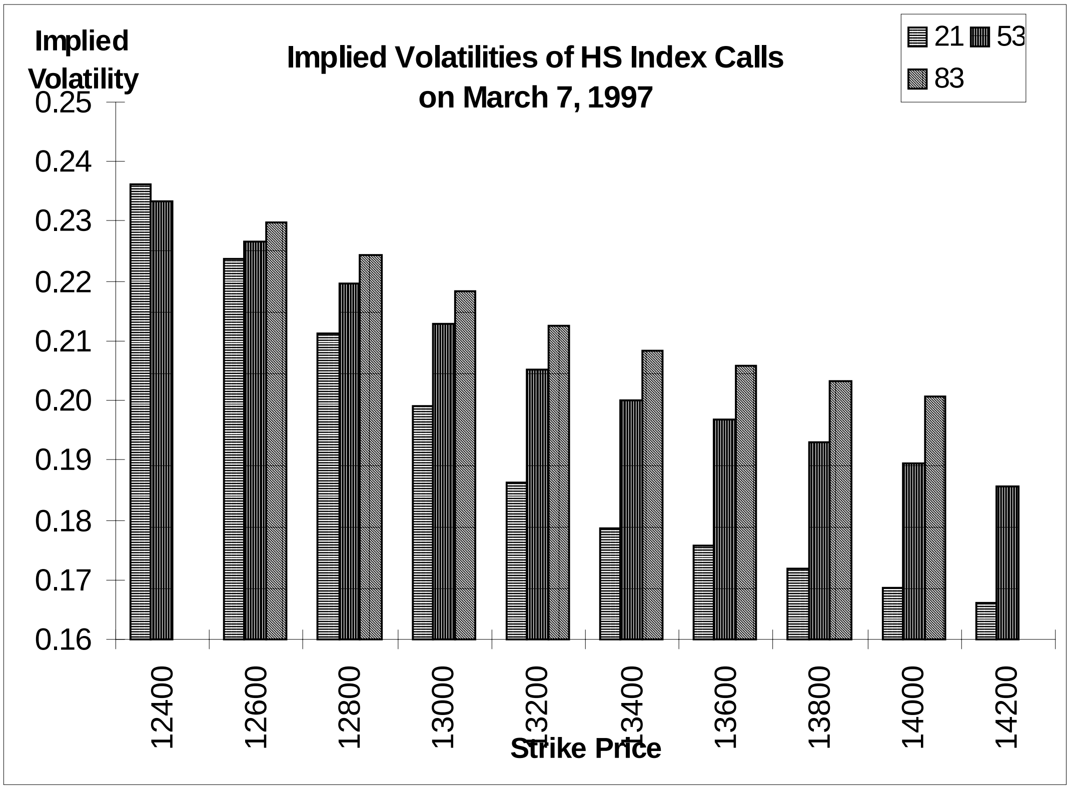
Example 1: FTSE 100 index options





Example 2: Hang Seng index options





2. GARCH Option Pricing Model (Duan, 1995)

- **Asset price dynamics modeled by a non-linear asymmetric GARCH (NGARCH)-in-mean process**

$$\ln\frac{S_{t+1}}{S_t} = r + \lambda\sigma_{t+1} - \frac{\sigma_{t+1}^2}{2} + \sigma_{t+1}\varepsilon_{t+1}$$

$$\sigma_{t+1}^2 = \beta_0 + \beta_1\sigma_t^2 + \beta_2\sigma_t^2(\varepsilon_t - \theta)^2$$

$$\varepsilon_{t+1}|F_t \overset{P}{\sim} N(0,1)$$

- **Locally risk-neutralized asset price process**

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{\sigma_{t+1}^2}{2} + \sigma_{t+1}\varepsilon_{t+1}^*$$
$$\sigma_{t+1}^2 = \beta_0 + \beta_1\sigma_t^2 + \beta_2\sigma_t^2(\varepsilon_t^* - \theta - \lambda)^2$$
$$\varepsilon_{t+1}^* | F_t \overset{Q}{\sim} N(0,1)$$

This implies

$$S_T = S_0 \exp\left[Tr - \frac{1}{2}\sum_{s=1}^{T}\sigma_s^2 + \sum_{s=1}^{T}\sigma_s\varepsilon_s^*\right]$$

- **Option pricing**

For a European call option with a payoff at time T:

$$Max(S_T - K, 0)$$

its time-0 value is

$$C(S_0, \sigma_1; K, T, r, \beta_0, \beta_1, \beta_2, \theta + \lambda) = e^{-rT} E_0^Q\{Max(S_T - K, 0)\}$$

- **Forex option pricing under GARCH**

Risk-neutralized exchange rate process under Black-Scholes

$$d\ln(e_t) = (r - r_f - \frac{\sigma^2}{2})dt + \sigma dW_t^*$$

Garman and Kohlhagen (1983) formula

$$C(e_0; K, T, r, r_f, \sigma)$$
$$= e_0 e^{-r_f T} N(d) - K e^{-rT} N(d_0 - \sigma\sqrt{T})$$

where

$$d = \frac{\ln \frac{e_0}{K} + (r - r_f + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Locally risk-neutralized exchange rate process under GARCH

$$\ln \frac{e_{t+1}}{e_t} = r - r_f - \frac{\sigma_{t+1}^2}{2} + \sigma_{t+1}\varepsilon_{t+1}^*$$
$$\sigma_{t+1}^2 = \beta_0 + \beta_1\sigma_t^2 + \beta_2\sigma_t^2(\varepsilon_t^* - \theta - \lambda)^2$$
$$\varepsilon_{t+1}^* | F_t \overset{Q}{\sim} N(0,1)$$

Forex option price under GARCH

$$C(e_0, \sigma_1; K, T, r, r_f, \beta_0, \beta_1, \beta_2, \theta + \lambda) = e^{-rT} E_0^Q \{Max(e_T - K, 0)\}$$

Note:

1. For quanto option pricing under GARCH, please see Duan and Wei (1999).

2. An arbitrage-free proof of the GARCH option pricing model can be found in Kallsen and Taqqu (1998). It is accomplished by using the geometric Brownian motion to connect the discrete-time GARCH model.

3. Numerical methods

- **Monte Carlo simulations**

1) Standard Monte Carlo simulation

$$S_t(i) = S_0 \exp[(r-\delta)t] Z_t(i)$$

$$Z_t(i) = Z_{t-1}(i) \exp[-0.5\sigma_t^2 + \sigma_t \varepsilon_t(i)]$$

2) Empirical martingale simulation (EMS) (Duan & Simonato, 1998)

$$S_t(i) = S_0 \exp[(r-\delta)t] Z_t(i)$$

$$Z_t(i) = \frac{Z_{t-1}(i) \exp[-0.5\sigma_t^2 + \sigma_t \varepsilon_t(i)]}{\frac{1}{n}\sum_{i=1}^{n} Z_{t-1}(i) \exp[-0.5\sigma_t^2 + \sigma_t \varepsilon_t(i)]}$$

Thus,

$$\frac{1}{n}\sum_{i=1}^{n} \exp[-(r-\delta)t] S_t(i) = S_0$$

Worksheet: Standard Monte Carlo Simulation for valuing call option under GARCH

Current stock price =		51	
Initial conditional s.d. (annualized) =		0.2	
Interest rate (annualized) =		0.05	
Number of sample paths =		10	
Maturity (days) =	2	Strike price (X) =	50

GARCH parameters:	
Beta0 =	1E-05
Beta1 =	0.8
Beta2 =	0.1
Theta =	0.5
Lambda =	0.3

Stationary standard deviations (annualized) implied by the GARCH parameters:

Data generating = 0.2206

Risk neutral = 0.3184

std. normal	std. normal
-0.8131	0.7647
-0.5470	0.5537
0.4109	0.0835
0.4370	-0.6313
0.5413	-0.1772
-1.0472	2.4048
0.3697	0.0706
-2.0435	-1.4961
-0.2428	-1.3760
0.3091	0.3845

S(0)	sd(1)	S(1)	sd(2)	S(2)	max(S(2)-X,0)
51	0.200	50.572	0.215	51.012	1.012
51	0.200	50.713	0.207	51.022	1.022
51	0.200	51.224	0.190	51.271	1.271
51	0.200	51.238	0.190	50.921	0.921
51	0.200	51.294	0.190	51.208	1.208
51	0.200	50.448	0.222	51.881	1.881
51	0.200	51.202	0.191	51.243	1.243
51	0.200	49.925	0.261	48.918	0.000
51	0.200	50.875	0.200	50.151	0.151
51	0.200	51.169	0.191	51.371	1.371

Discounted average (or Monte Carlo price) = 1.0079

Worksheet: Empirical Martingale Simulation for valuing call option under GARCH

Current stock price =		51	
Initial conditional s.d. (annualized) =		0.2	
Interest rate (annualized) =		0.05	
Number of sample paths =		10	
Maturity (days) =	2	Strike price (X) =	50

GARCH parameters:	
Beta0 =	1E-05
Beta1 =	0.8
Beta2 =	0.1
Theta =	0.5
Lambda =	0.3

Stationary standard deviations (annualized) implied by the GARCH parameters:
Data generating = 0.2206
Risk neutral = 0.3184

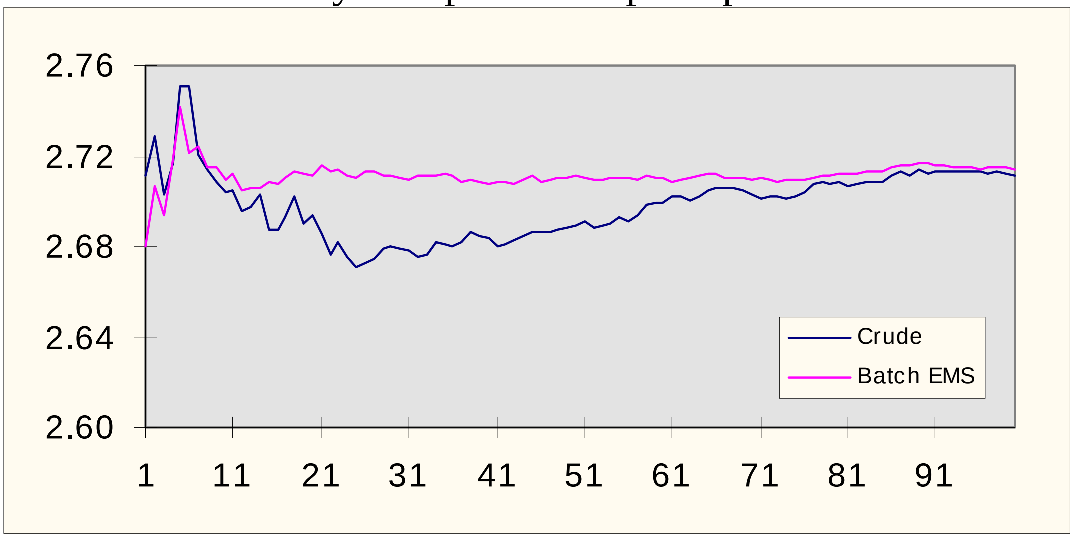
std. normal	std. normal
-0.8131	0.7647
-0.5470	0.5537
0.4109	0.0835
0.4370	-0.6313
0.5413	-0.1772
-1.0472	2.4048
0.3697	0.0706
-2.0435	-1.4961
-0.2428	-1.3760
0.3091	0.3845

S(0)	sd(1)	S(1)	S*(1)	sd(2)	S(2)	S*(2)	max(S*(2)-X,0)
51	0.200	50.572	50.712	0.215	51.012	51.126	1.126
51	0.200	50.713	50.854	0.207	51.022	51.137	1.137
51	0.200	51.224	51.366	0.190	51.271	51.386	1.386
51	0.200	51.238	51.380	0.190	50.921	51.036	1.036
51	0.200	51.294	51.436	0.190	51.208	51.323	1.323
51	0.200	50.448	50.588	0.222	51.881	51.998	1.998
51	0.200	51.202	51.344	0.191	51.243	51.357	1.357
51	0.200	49.925	50.063	0.261	48.918	49.027	0.000
51	0.200	50.875	51.016	0.200	50.151	50.264	0.264
51	0.200	51.169	51.311	0.191	51.371	51.486	1.486
Average		50.866			50.900		

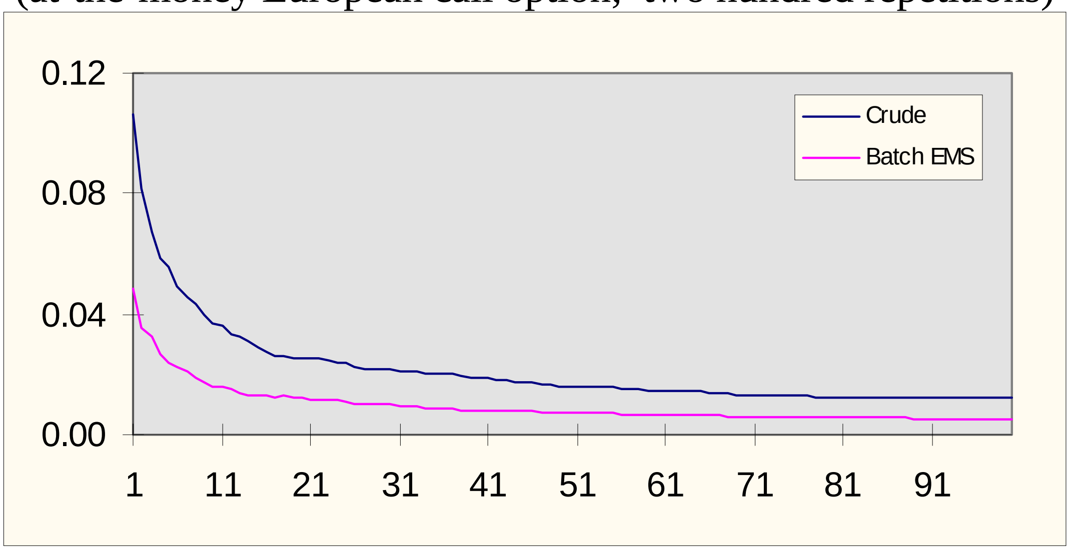
Discounted average (or Monte Carlo price) = 1.1109

A comparison: Black-Scholes option pricing model (batch size = 1000; S_0=100, r=0.1, δ=0, σ=0.2, T=1 month)

At-the-money European call option price estimate



Standard deviation of the price estimate
(at-the-money European call option; two hundred repetitions)



- **Markov chain approximation** (Duan & Simonato, 1999)

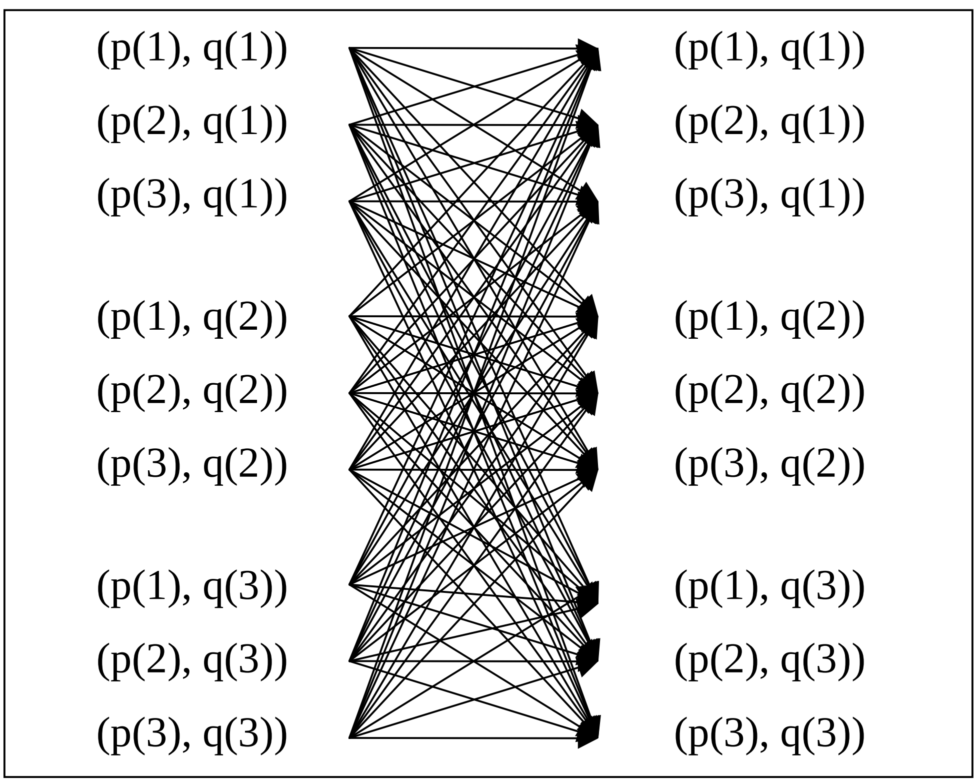
Discretize the underlying asset prices

$$p_t = -(r - \frac{1}{2}h^*)t + \ln(S_t)$$
$$q_{t+1} = \ln(h_{t+1})$$

where

$$h^* = \frac{\beta_0}{1 - \beta_1 - \beta_2[1 + (\theta + \lambda)^2]}$$

Example: m=3, $n=3$



Transition probability matrix

$$\Pi = \begin{bmatrix} \pi(1,1;1,1) & \pi(1,1;2,1) & \cdots & \pi(1,1;m,1) & \pi(1,1;1,2) & \cdots & \pi(1,1;m,n) \\ \pi(2,1;1,1) & \pi(2,1;2,1) & \cdots & \pi(2,1;m,1) & \pi(2,1;1,2) & \cdots & \pi(2,1;m,n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi(m,1;1,1) & \pi(m,1;2,1) & \cdots & \pi(m,1;m,1) & \pi(m,1;1,2) & \cdots & \pi(1,2;m,n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

Since

$$\ln\frac{S_{t+1}}{S_t} = r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1}$$

$$h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1}(\varepsilon_{t+1} - \theta - \lambda)^2$$

q_{t+2} is a deterministic function of q_{t+1}, p_{t+1}, p_t;
that is, $q_{t+2} = \Phi(q_{t+1}, p_{t+1}, p_t)$

$$\pi(i,j;k,l) =$$

$$\begin{cases} \Pr^Q\{p_{t+1} \in C(k) \mid p_t = p(i), q_{t+1} = q(j)\} & \text{if } \Phi\big(q(j), p(k), p(i)\big) \in D(l) \\ 0 & \text{otherwise} \end{cases}$$

American option prices in the GARCH framework

Price vector

$$\bar{P}' = [p(1) \quad p(2) \quad \cdots \quad p(m) \quad \cdots \quad p(1) \quad p(2) \quad \cdots \quad p(m)]$$

American option price computation

$$\bar{V}(\bar{P}, t) = \max\{g(\bar{P}, K, t), e^{-r}\Pi\bar{V}(\bar{P}, t+1)\}$$

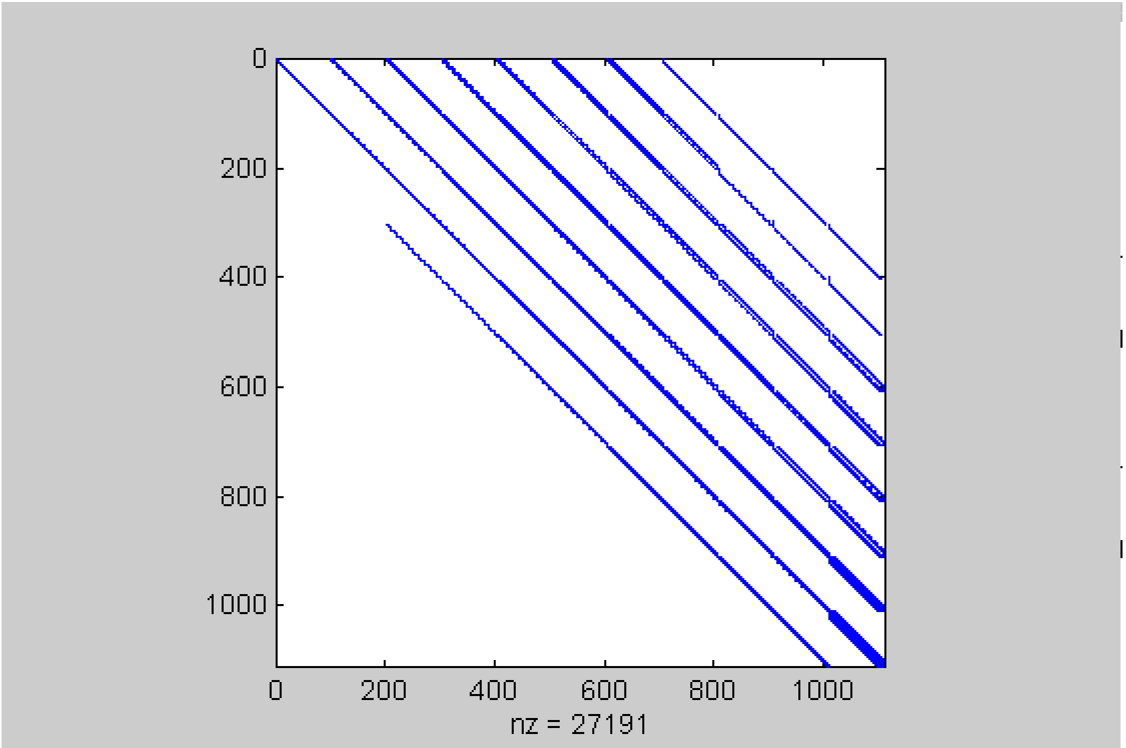
where

$\bar{V}(\bar{P}, t)$: American option's price
K : Strike price
$g(\bar{P}, K, t)$: European option's payoff function
$\bar{V}(\bar{P}, T) = g(\bar{P}, K, T)$

Density of the transition probability matrix (GARCH)

m=101, n=11,

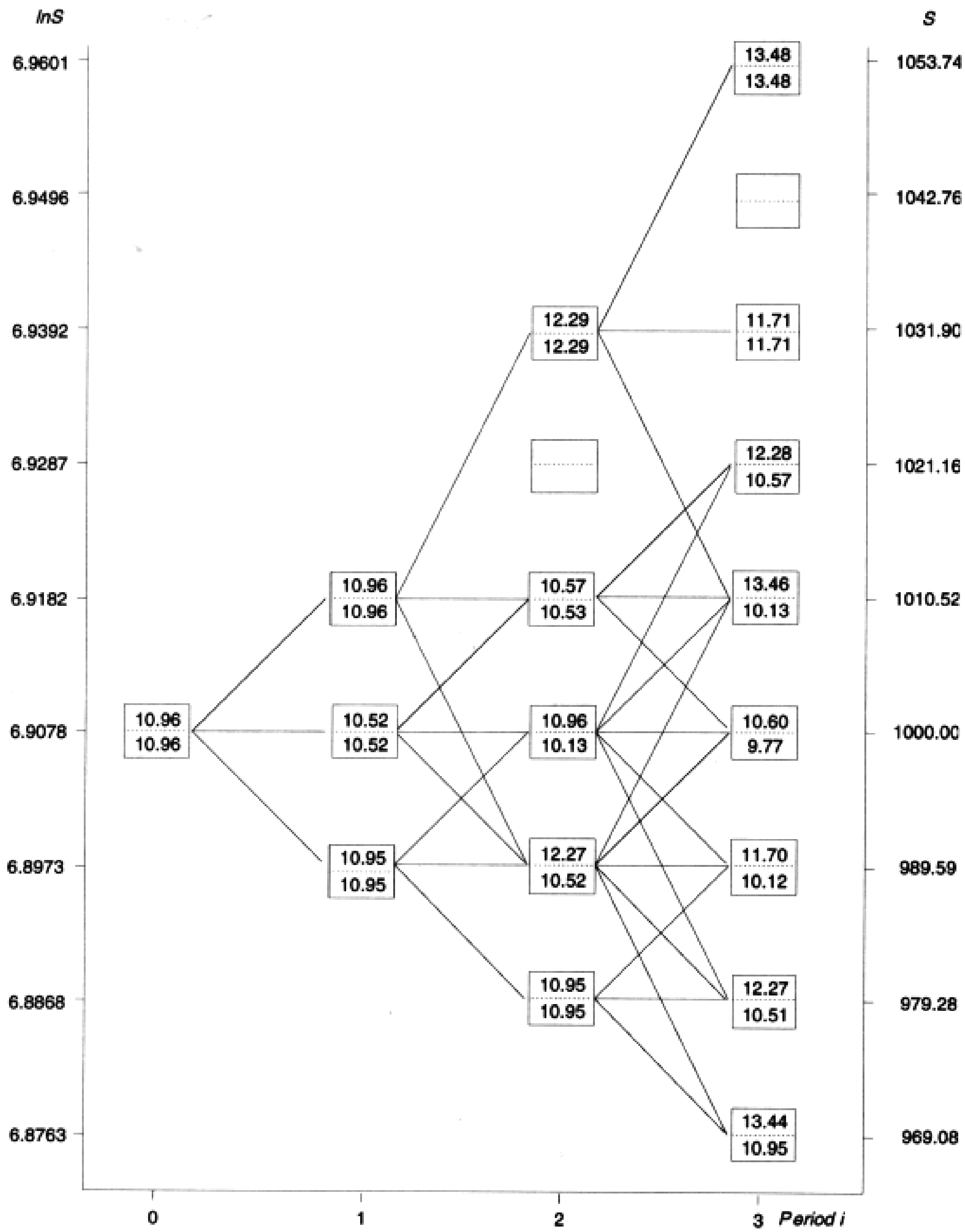
$$S_0 = 50, r = 0.05, \beta_0 = 0.00001, \beta_1 = 0.8, \beta_2 = 0.1, \lambda = 0.2$$



- **Lattice construction** (Ritchken & Trevor, 1999)

(The following three figures are taken from Ritchken & Trevor, 1999)

Figure 1: Lattice of State Variables over Three Days*



Note: The maximum and minimum conditional volatilities are given in the boxes. These figures are multiplied by 10^5. S_0=1000, r=0, λ=0,β_0=0.0000065, β_1=0.9,β_2=0.04,c=0.

Figure 2: Illustrative Lattice for Three-Period At-the-Money Call Option*

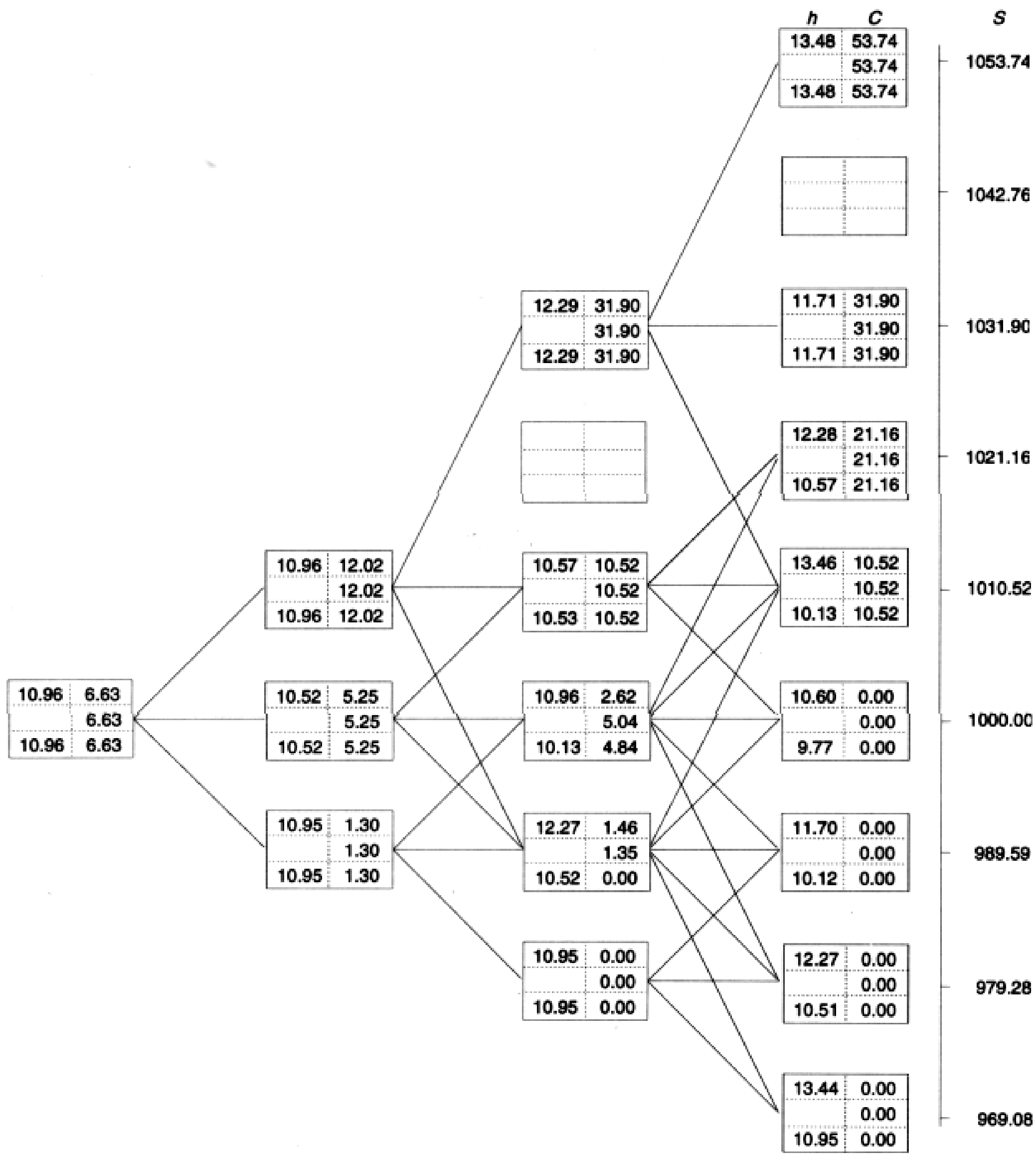
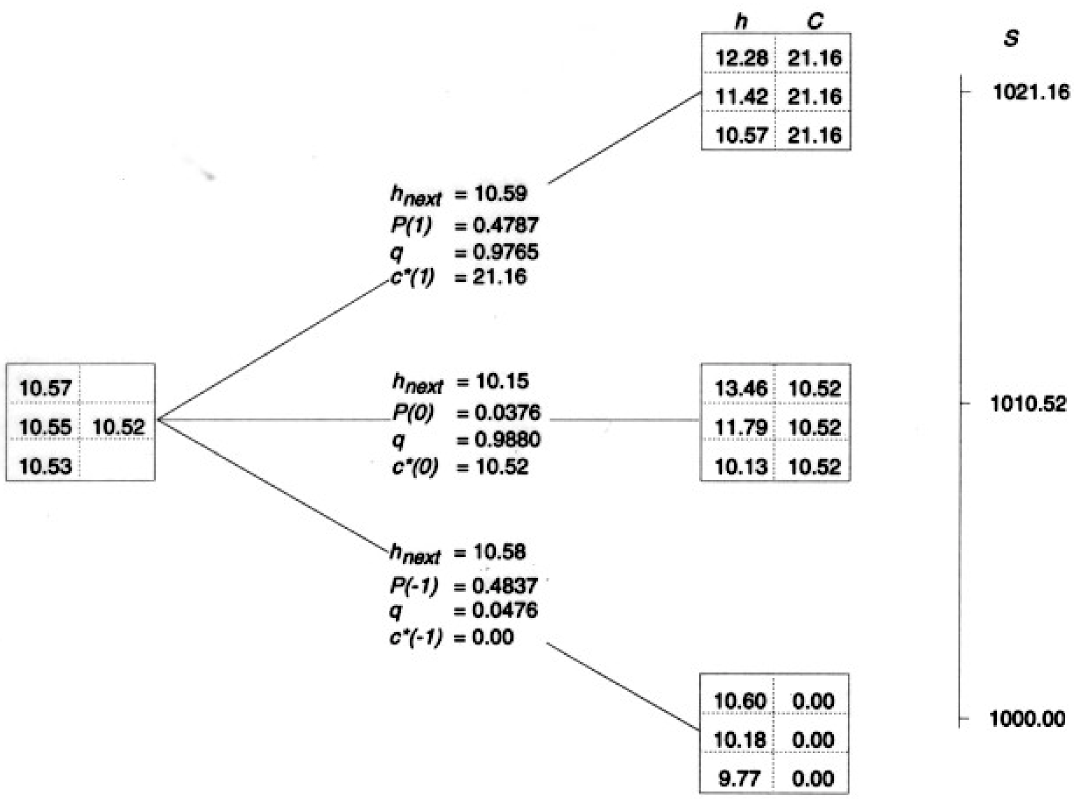


Figure 3: Illustration of Lattice Computation*



- **Analytical approximation** (Duan, Gauthier & Simonato, 1999)

For a European <u>call</u> option payoff at time T,

$$Max(S_T - K, 0)$$

its time-0 value is by the approximate closed-form solution

$$C_{app} = C + \kappa_3 A_3 + (\kappa_4 - 3) A_4$$

where

$$C = S_0 N(\tilde{d}) - Ke^{-rT} N(\tilde{d} - \sigma_{\rho_T})$$

$$\tilde{d} = d + \delta$$

$$d = \frac{\ln(\frac{S_0}{K}) + rT + \frac{1}{2}\sigma_{\rho_T}^2}{\sigma_{\rho_T}}$$

$$\delta = \frac{\mu_{\rho_T} - rT + \frac{1}{2}\sigma_{\rho_T}^2}{\sigma_{\rho_T}}$$

$$A_3 = \frac{1}{3!} S_0 \sigma_{\rho_T} [(2\sigma_{\rho_T} - \tilde{d}) n(\tilde{d}) - \sigma_{\rho_T}^2 N(\tilde{d})]$$

$$A_4 = \frac{1}{4!} S_0 \sigma_{\rho_T} [(\tilde{d}^2 - 1 - 3\sigma_{\rho_T}(\tilde{d} - \sigma_{\rho_T})) n(\tilde{d}) - \sigma_{\rho_T}^3 N(\tilde{d})]$$

Note: μ_{ρ_T} and σ_{ρ_T} are the mean and standard deviation of the cumulative return, i.e., $\ln \frac{S_T}{S_0}$, conditional on time-0 information; κ_3 and κ_4 are the skewness and kurtosis coefficients of the standardized cumulative return, conditional on time-0 information.

Conditional moments of the cumulative return

Figure 1: Standardized first moment of the cumulative return

(GARCH parameter values : $\beta_0 = 0.00001, \beta_1 = 0.7, \beta_2 = 0.1, \lambda + \theta = 1.0, h_1 = h^*$ **)**

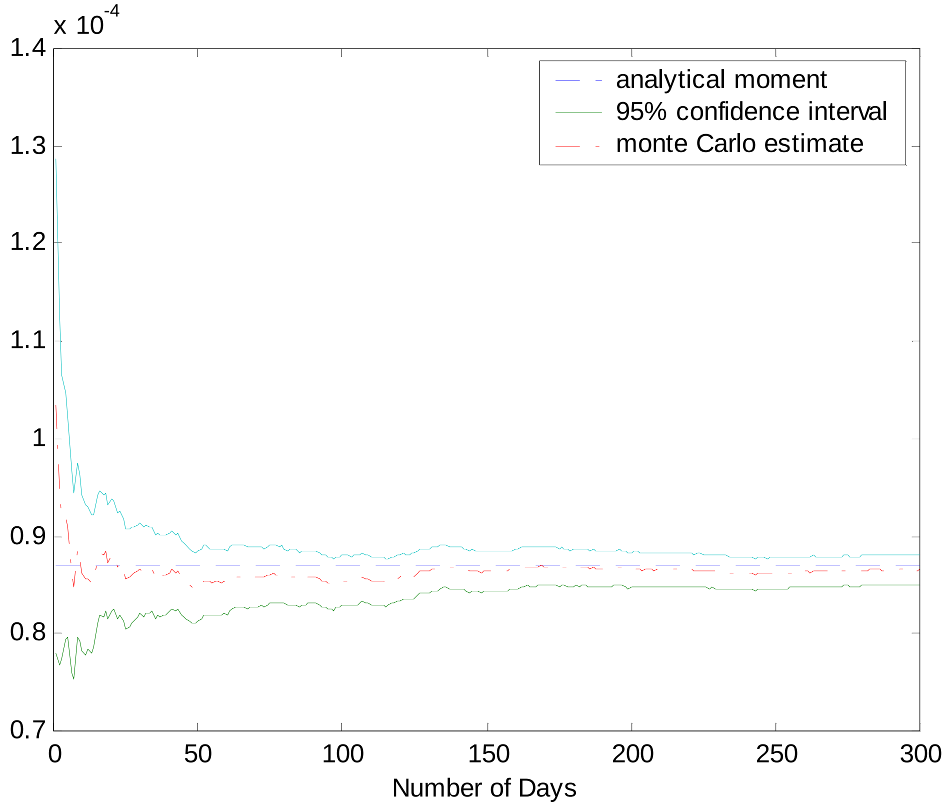


Figure 2 : Standardized second moment of the cumulative return

(GARCH parameter values : $\beta_0 = 0.00001, \beta_1 = 0.7, \beta_2 = 0.1, \lambda + \theta = 1.0, h_1 = h^*$ **)**

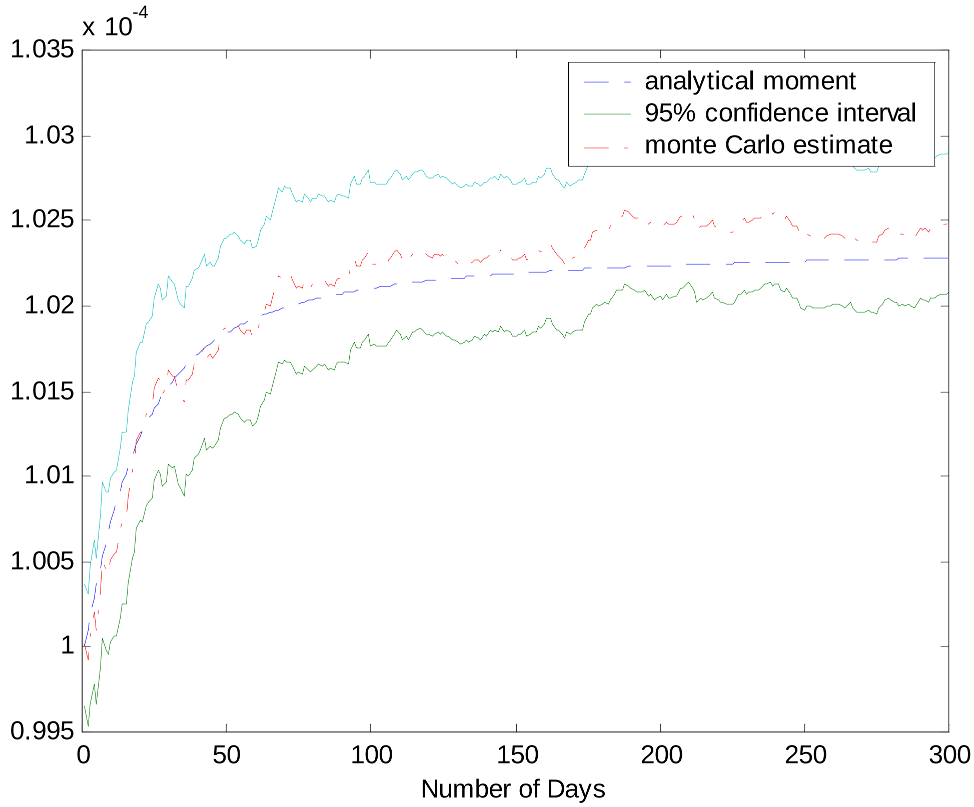


Figure 3 : Skewness of the standardized cumulative return

(GARCH parameter values : $\beta_0 = 0.00001, \beta_1 = 0.7, \beta_2 = 0.1, \lambda + \theta = 1.0, h_1 = h^*$ **)**

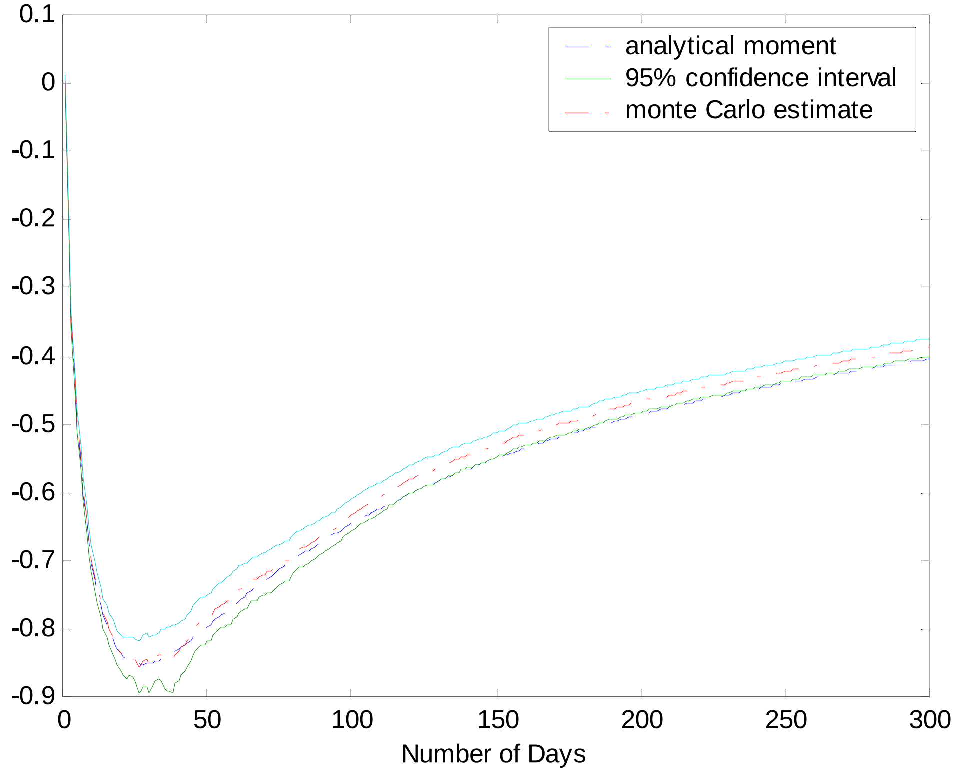
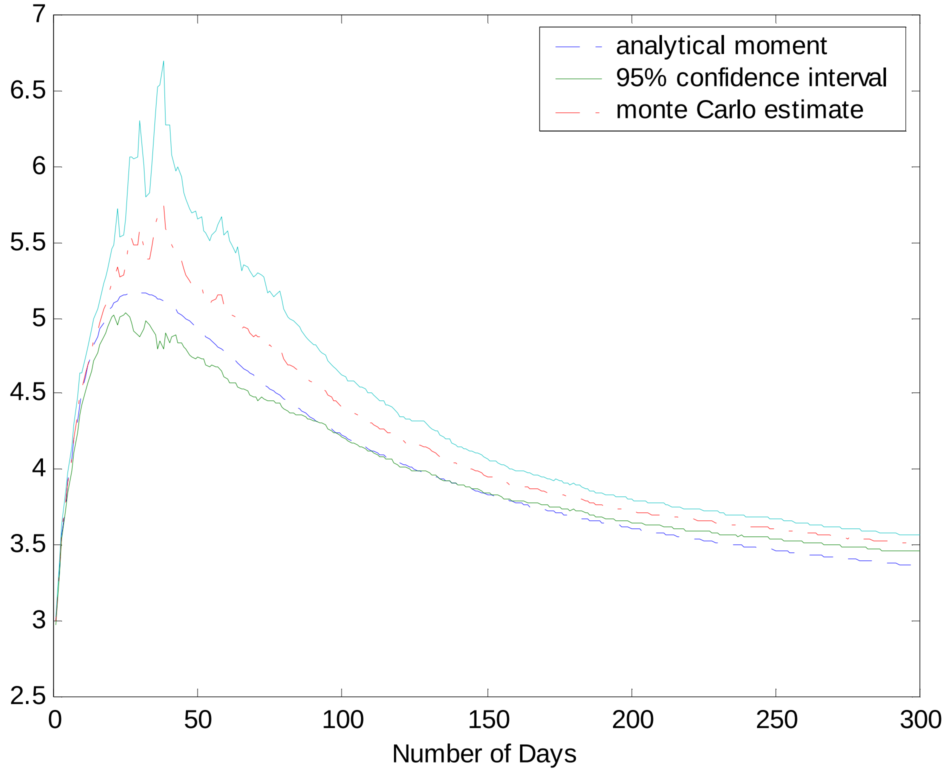


Figure 4 : Kurtosis of the standardized cumulative return

(GARCH parameter values : $\beta_0 = 0.00001, \beta_1 = 0.7, \beta_2 = 0.1, \lambda + \theta = 1.0, h_1 = h^*$ **)**



- **Neural network approximation** (Hanke, 1997)

Euro Call Option under GARCH

S=	50.5		
X=	50		
Mat =	25	days	

	Lambda	c-sig/sig	S/X	Mat (yrs)	sigma	beta2	beta1	r
Inputs	0	1	1.01	0.0685	0.2	0.1	0.8	0.05
Weights	0.214	0.417	1.713	-2.851	-1.799	-0.683	-0.467	-1.011
(layer 1)	-0.033	0.161	0.415	0.360	-0.358	0.394	-0.504	-0.244
	-0.005	0.452	1.979	0.314	0.077	-0.177	0.005	0.134
	-0.502	-0.059	-2.166	-0.796	-0.144	-0.040	0.059	0.213
	-0.726	0.097	0.192	1.500	-0.147	-0.275	0.231	-1.078
	-0.016	-0.025	3.292	3.340	-0.432	0.022	0.034	0.266
	-0.443	-0.093	1.353	0.587	0.071	0.020	0.129	0.170
	-0.209	0.041	-14.132	0.345	0.361	-0.040	-0.017	-0.164
	-0.087	-0.077	0.316	3.570	-0.994	0.068	-0.062	0.356
	0.289	-0.025	3.680	3.649	-0.740	0.034	0.032	0.268
	-0.296	-0.165	5.843	-5.310	0.006	-1.393	-0.376	-2.993
	0.051	-0.771	2.812	-0.163	0.612	1.049	1.080	-0.197
	0.136	0.065	2.504	2.733	-1.459	-0.225	-0.255	0.623
	0.130	-0.860	-1.722	-0.063	0.206	0.339	1.247	-0.367
	0.632	0.022	-14.877	-1.672	-0.158	0.014	-0.008	-0.116
	0.034	0.440	-1.198	0.658	1.195	1.419	-0.667	0.044
	-0.045	0.751	-1.830	-0.414	-0.589	-0.987	-1.009	0.300
	0.420	-0.309	-1.273	4.204	1.422	0.453	0.302	0.631
	-0.800	-0.051	3.899	0.316	-1.470	0.150	0.093	0.255
	0.372	0.132	-1.663	2.097	1.040	-0.253	-0.155	-0.221
	0.159	-0.070	-3.087	2.330	1.479	0.054	0.071	-0.617
	-0.034	0.006	6.028	2.103	0.046	-0.045	-0.052	0.183
	0.522	-0.050	-2.949	-3.214	0.021	0.036	0.063	0.430
	0.623	0.024	-1.092	-1.101	-0.301	-0.332	-0.121	-0.447

Product	Biases	tanh(P+B)	Weights (layer 2)	Product
1.100	-0.236	0.698	0.094	0.066
0.157	-0.895	-0.628	0.003	-0.002
2.481	-1.694	0.656	-0.047	-0.031
-2.276	4.288	0.965	-1.717	-1.657
0.468	-0.949	-0.447	0.008	-0.003
3.485	-3.034	0.422	-1.651	-0.697
1.443	-1.464	-0.022	0.683	-0.015
-14.162	13.952	-0.207	0.131	-0.027
0.263	1.526	0.946	0.387	0.366
3.836	-3.283	0.503	1.323	0.666
4.784	-6.767	-0.963	0.016	-0.015
3.140	-5.557	-0.984	0.509	-0.501
2.294	-3.253	-0.744	0.185	-0.138
-1.549	1.970	0.398	0.001	0.000
-15.161	15.023	-0.137	-0.120	0.017
-0.876	0.771	-0.104	0.004	0.000
-2.135	4.251	0.971	0.524	0.509
-0.704	1.525	0.675	-0.170	-0.115
3.716	-3.130	0.527	-0.252	-0.133
-1.356	-1.377	-0.992	1.632	-1.619
-2.701	3.921	0.840	0.202	0.169
6.210	-6.547	-0.324	0.126	-0.041
-3.169	5.366	0.976	1.064	1.038
-1.366	1.301	-0.066	-0.328	0.022
Sum				-2.1422
Bias (2nd layer)			+	2.1702
Option price (before adjustment)				0.0280
Option price (after adjustment)				1.3996

Note: The above spreadsheet is based on the network constructed by Hanke with one hidden layer of 24 neurons. This network is limited to a range of parameters as follows: $0 \le \lambda \le 0.001$, $0.8 \le \sigma_1/\sigma \le 1.2$, $0.7 \le S/X \le 1.3$, $1/365 \le \tau \le 30/365$, $0.1 \le \sigma \le 0.4$, $0.5 \le \beta_1 \le 0.8$, $0.1 \le \beta_2 \le 0.3$, $0 \le r \le 0.1$.

4. Tackling volatility smile using GARCH

- **Put-call parity regression**

$$C(\tau, X) - P(\tau, X) = S(\tau) - X\exp[-\tau r(\tau)] + \varepsilon(\tau, X)$$

This regression can be run for every τ to obtain an intercept and slope. The intercept is $S(\tau)$ and the slope is $-\exp[-\tau r(\tau)]$, which implies $r(\tau)$.

Constrained put-call parity regression:

Make certain that $S(\tau)$ is a non-increasing function of τ.

Let τ_1 be the smallest maturity and use the following formulation:

$$C(\tau, X) - P(\tau, X) = S(\tau_1) - a(\tau) - X\exp[-\tau r(\tau)] + \varepsilon(\tau, X)$$

for $\tau = \tau_1, \tau_2, \cdots, \tau_n$

where

$a(\tau) \geq 0$ and $a(\tau_1) = 0$

FTSE 100 index options (Euro-style) on March 26, 1997

Maturity	23	23	51	51	86	86	177	177	268	268
Strike price	Call	Put	Call	Put	Call	Put	Call	Put	Call	Put
4125	179.5	11.5	217.5	38.0	245.5	58.5	302.5	94.5	364	119.5
4175	136.0	17.0	179.0	49.0	209.5	71.0				
4225	96.0	27.0	143.0	63.0	174.5	85.5	235.5	124.5	297	148.5
4275	62.5	43.0	111.0	80.0	142.0	102.5				
4325	36.0	66.5	83.0	102.0	113.0	123.0	175.0	161.5	236.5	184
4375	18.0	98.5	59.0	127.5	88.0	147.0				
4425	8.0	137.5	40.0	158.5	68.5	177.0	126.0	210.0	183	226.5
4475	3.0	182.5	26.0	193.5	49.0	207.0				

Maturity	23	51	86	177	268
Strike price	C-P	C-P	C-P	C-P	C-P
4125	168.0	179.5	187.0	208.0	244.5
4175	119.0	130.0	138.5		
4225	69.0	80.0	89.0	111.0	148.5
4275	19.5	31.0	39.5		
4325	-30.5	-19.0	-10.0	13.5	52.5
4375	-80.5	-68.5	-59.0		
4425	-129.5	-118.5	-108.5	-84.0	-43.5
4475	-179.5	-167.5	-158.0		

Intercept	4267.3	4272.1	4257.0	4223.8	4204.5
Slope	-0.9937	-0.9921	-0.9865	-0.9735	-0.96
R square	1	1	1	1	1

Imp. index	4267.3	4272.1	4257.0	4223.8	4204.5
Int. rate	10.04%	5.65%	5.75%	5.54%	5.56%

Constrained regressions

Imp. index	4269.7	4269.7	4257.0	4223.9	4204.5
Int. rate	9.16%	6.05%	5.75%	5.54%	5.56%

- **FTSE 100 index options**

A) Calibrate the GARCH option pricing model using the market prices of traded options in three steps.

<u>Step 1</u>: Estimate implied interest rates and index values using the put-call parity regression for March 26, 1997

Implied interest rates:

0.091591	0.060473	0.057472	0.055374	0.055604

Implied index spots:

4269.69	4269.69	4256.98	4223.86	4204.48

<u>Step 2</u>: Compute market implied volatilities for calls

			Maturities		
	23	51	86	177	268
Strike Price					
4125	0.148192	0.167101	0.162538	0.156996	0.158193
4175	0.138595	0.161283	0.158904		
4225	0.129007	0.154893	0.153415	0.150791	0.152135
4275	0.122565	0.149574	0.147791		
4325	0.115908	0.144424	0.142836	0.143619	0.146566
4375	0.110632	0.138826	0.138783		
4425	0.108071	0.134058	0.137396	0.138915	0.141300
4475	0.105673	0.130516	0.131567		

Step 3: Calibrate the GARCH model to minimize the difference between the market implied volatility and the model implied volatility.

Estimated parameter values:

β_0 = 0.00000429

β_1 = 0.72507034

β_2 = 0.07560027

$\theta+\lambda$ = 1.35643575

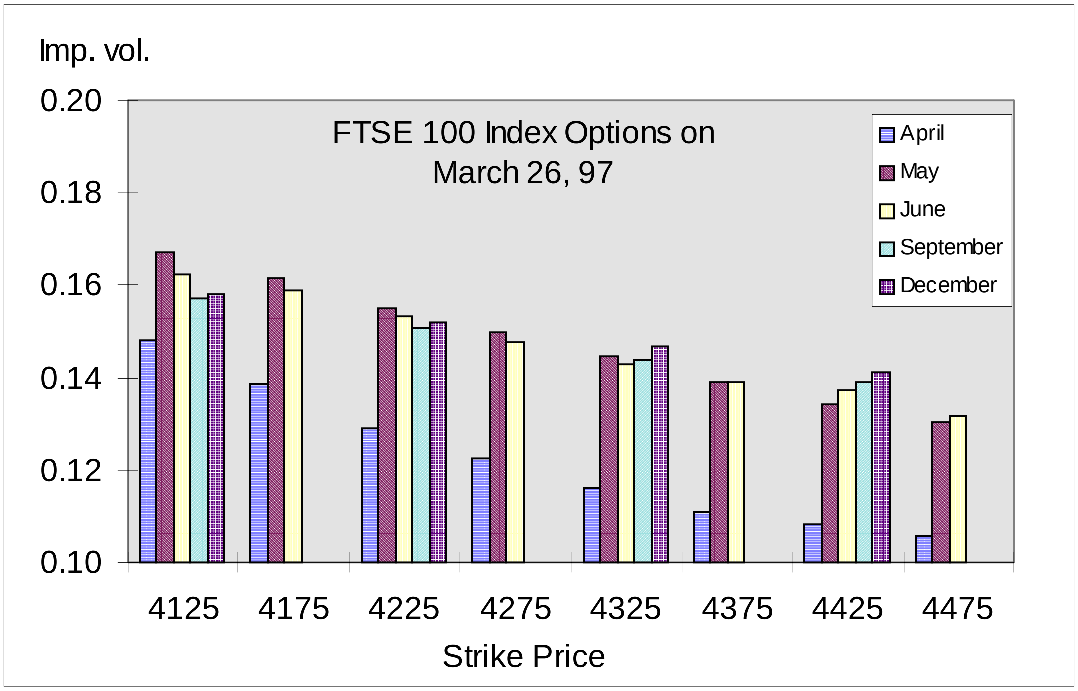
σ_1 = 0.09889376

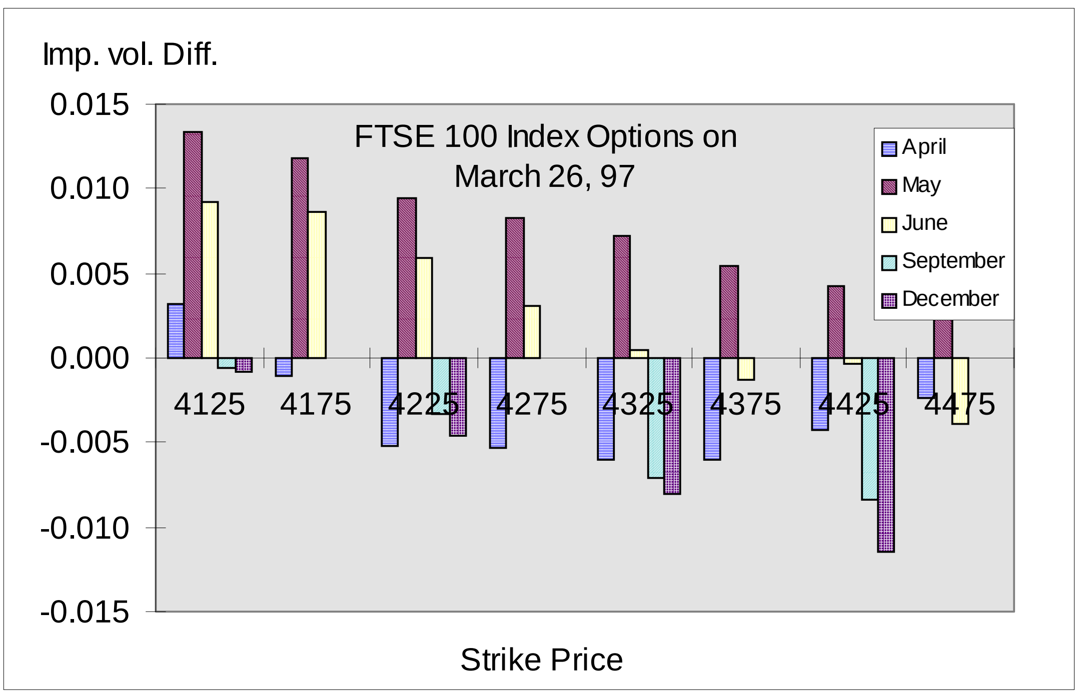
Imp. Risk-Neutral Stationary S.D. = 16.12%

Implied volatilities derived from the GARCH option pricing model

	Maturities				
Strike Price	23	51	86	177	268
4125	0.144981	0.153777	0.153322	0.157534	0.158991
4175	0.139704	0.149455	0.150253	0.155788	0.157867
4225	0.134155	0.145439	0.147458	0.154096	0.156758
4275	0.127844	0.141323	0.144724	0.152474	0.155685
4325	0.121891	0.137218	0.142327	0.150752	0.154627
4375	0.116696	0.133375	0.140043	0.148963	0.153705
4425	0.112325	0.129775	0.137806	0.147262	0.152766
4475	0.108033	0.126464	0.135494	0.145531	0.151839

Root Mean Squared Error = 0.00643679


Imp. vol.
0.20
0.18
0.16
0.14
0.12
0.10
FTSE 100 Index Options on
March 26, 97
April
May
June
September
December
4125
4175
4225
4275
4325
4375
4425
4475
Strike Price


Imp. vol. Diff.
0.015
0.010
0.005
0.000
-0.005
-0.010
-0.015
FTSE 100 Index Options on
March 26, 97
April
May
June
September
December
4125
4175
4225
4275
4325
4375
4425
4475
Strike Price

B) Out-sample performance on April 2, 1997 (1 week later)

Step 1: Estimate implied interest rates and index values using the put-call parity regression

Implied interest rates:

0.087787	0.055221	0.053111	0.054358	0.058546

Implied index spots:

4215.80	4215.80	4204.43	4170.63	4140.97

Step 2: Compute market implied volatilities for calls

	Maturities				
	16	44	79	170	261
Strike Price					
4075	0.185401	0.184434	0.176989		
4125	0.171461	0.177604	0.170641	0.164747	0.163311
4175	0.160283	0.170957	0.165924		
4225	0.151814	0.165289	0.160954	0.158433	0.157141
4275	0.142793	0.158884	0.156038		
4325	0.137634	0.153740	0.151518	0.151709	0.151619
4375	0.131314	0.148823	0.147214		
4425	0.130085	0.143318	0.143092	0.144766	0.147019

Step 3: Calibrate the GARCH model to minimize the difference between the market implied volatility and the model implied volatility.

Pre-set parameter values:

β_0 = 0.00000429

β_1 = 0.72507034

β_2 = 0.07560027

$\theta+\lambda$ = 1.35643575

Estimated parameter values:

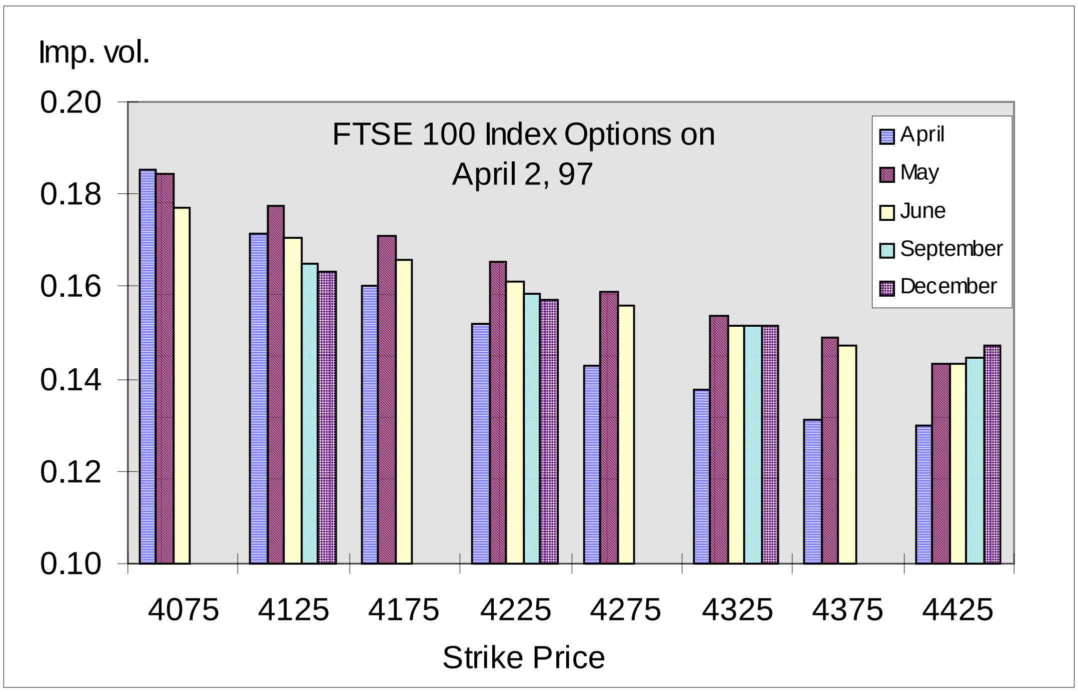
σ_1 = 0.16876672

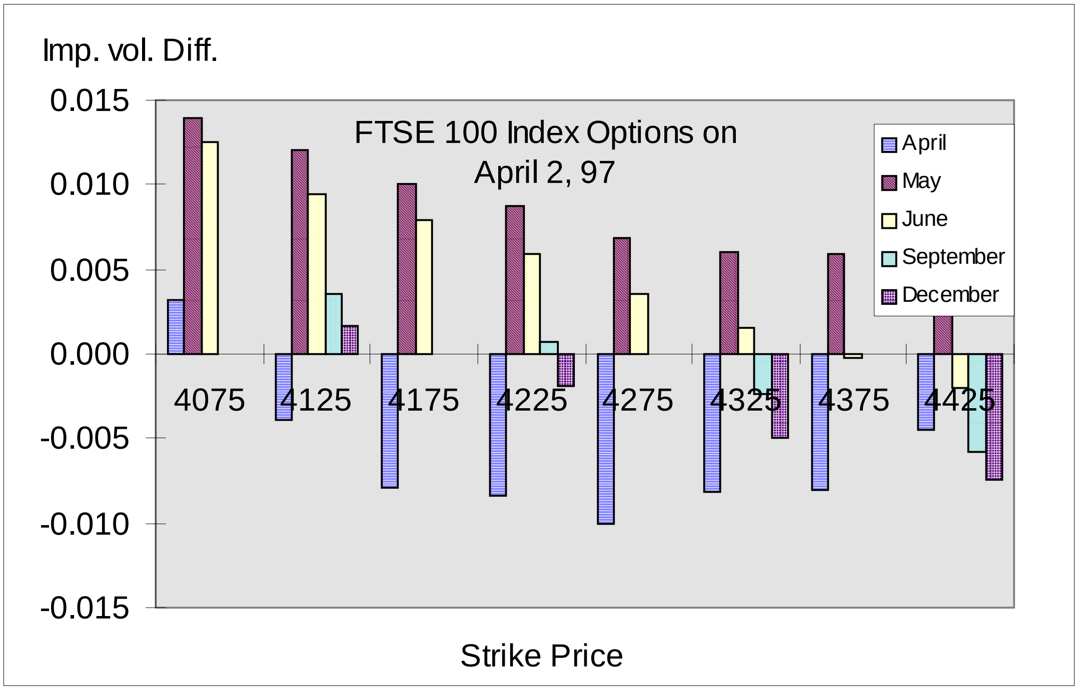
Imp. Risk-Neutral Stationary S.D. = 16.12% (unchanged)

Implied volatilities derived from the GARCH option pricing model

	Maturities				
	16	44	79	170	261
Strike Price					
4075	0.182192	0.170503	0.164460	0.163236	0.163042
4125	0.175398	0.165596	0.161139	0.161253	0.161672
4175	0.168165	0.160894	0.157961	0.159455	0.160368
4225	0.160163	0.156524	0.155017	0.157713	0.159062
4275	0.152872	0.152044	0.152459	0.155927	0.157798
4325	0.145817	0.147760	0.150011	0.154032	0.156564
4375	0.139310	0.142900	0.147438	0.152296	0.155454
4425	0.134518	0.138626	0.145090	0.150521	0.154428

Root Mean Squared Error = 0.00699941





C) Out-sample performance on April 9, 1997 (2 weeks later)

Pre-set parameter values:

β_0 = 0.00000429

β_1 = 0.72507034

β_2 = 0.07560027

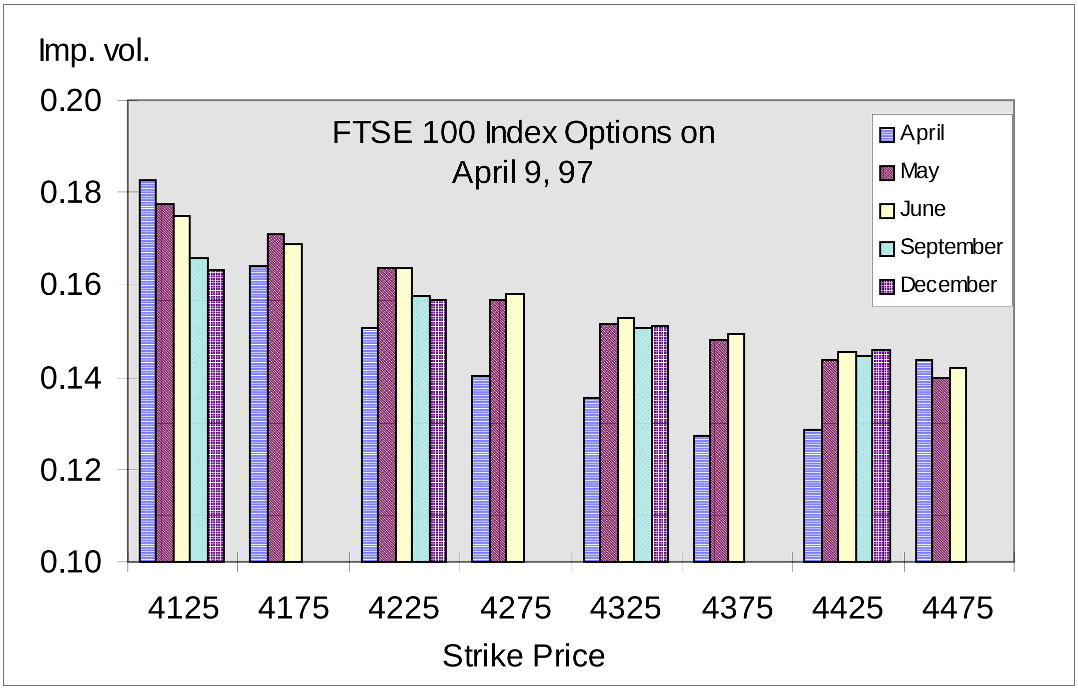
$\theta+\lambda$ = 1.35643575

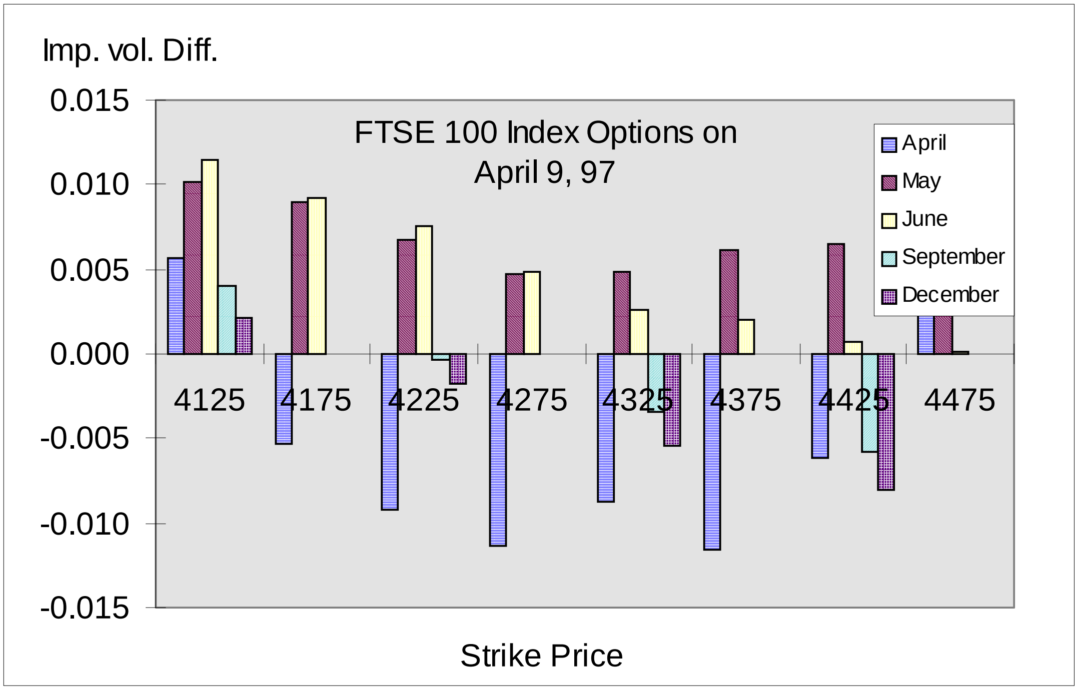
Estimated parameter values:

σ_1 = 0.15313167

Imp. Risk-Neutral Stationary S.D. = 16.12% (unchanged)

Root Mean Squared Error = 0.00697208


Imp. vol.
FTSE 100 Index Options on
April 9, 97
April
May
June
September
December
0.20
0.18
0.16
0.14
0.12
0.10
4125
4175
4225
4275
4325
4375
4425
4475
Strike Price


Imp. vol. Diff.
FTSE 100 Index Options on
April 9, 97
April
May
June
September
December
0.015
0.010
0.005
0.000
-0.005
-0.010
-0.015
4125
4175
4225
4275
4325
4375
4425
4475
Strike Price

5. The general properties of the GARCH option pricing model

- **Leptokurtic (fat-tailed) and skewed distributions**

Under the data generating probability

$$E(\varepsilon_t^4) = 3\frac{1-v^2}{1-u} > 3$$

$$Cov(\varepsilon_t, \sigma_{t+1}^2) = -\frac{2\theta\beta_0\beta_2}{1-\beta_1-\beta_2(1+\theta^2)}$$

where

$$u = \beta_2^2(3+6\theta^2+\theta^4)+2\beta_1\beta_2(1+\theta^2)+\beta_1^2$$

$$v = \beta_2(1+\theta^2)+\beta_1$$

Under the risk-neutralized probability

$$E^Q(\varepsilon_t^{*4}) = 3\frac{1-v^{*2}}{1-u^*} > 3$$

$$Cov^Q(\varepsilon_t^*, \sigma_{t+1}^2) = -\frac{2(\theta+\lambda)\beta_0\beta_2}{1-\beta_1-\beta_2[1+(\theta+\lambda)^2]}$$

$$u^* = \beta_2^2[3+6(\theta+\lambda)^2+(\theta+\lambda)^4]+2\beta_1\beta_2[1+(\theta+\lambda)^2]+\beta_1^2$$

$$v^* = \beta_2[1+(\theta+\lambda)^2]+\beta_1$$

Note:

1. θ is expected to be positive for equity returns
2. λ is expected to be positive for equity returns

<u>Conclusions</u>:

1) Under the data generating probability

- $\ln \frac{S_T}{S_0}$ is negatively skewed if $\theta > 0$
- $\ln \frac{S_T}{S_0}$ is leptokurtic

2) Under the risk-neutral probability

- $\ln \frac{S_T}{S_0}$ is negatively skewed if $\theta + \lambda > 0$
- $\ln \frac{S_T}{S_0}$ is leptokurtic

Prediction for implied volatilities (or option prices):

A downward skewed implied volatility smile

- **Risk premium and its effect on stationary (long-run) volatility**

Physical long-run volatility:

$$\sigma = \sqrt{\frac{\beta_0}{1 - \beta_1 - \beta_2(1 + \theta^2)}}$$

Risk-neutralized long-run volatility:

$$\sigma^* = \sqrt{\frac{\beta_0}{1 - \beta_1 - \beta_2[1 + (\theta + \lambda)^2]}}$$

Note: <u>σ^* is greater than σ if λ and θ share the same sign.</u>

Prediction for implied volatilities (option prices):

Implied volatility should be higher than "realized" or "historical" volatility.

6. Utilize the volatility smile in applications

- **Pricing exotic options**

On March 26, 1997, one can price exotic options, say a lookback call, using the estimated parameter values:

β_0 = 0.00000429
β_1 = 0.72507034
β_2 = 0.07560027
$\theta+\lambda$ = 1.35643575
σ_1 = 0.09889376

Worksheet: EMS for valuing a lookback call option under GARCH

Current stock price =	51
Initial conditional s.d. (annualized) =	0.0989
Interest rate (annualized) =	0.05
Number of sample paths =	10
Maturity (days) =	2

GARCH parameters:	
Beta0 =	4E-06
Beta1 =	0.7251
Beta2 =	0.0756
Theta =	1.3564
Lambda =	0

Stationary standard deviations (annualized) implied by the GARCH parameters:
Data generating = 0.1612
Risk neutral = 0.1612

std. normal	std. normal
-0.8131	0.7647
-0.5470	0.5537
0.4109	0.0835
0.4370	-0.6313
0.5413	-0.1772
-1.0472	2.4048
0.3697	0.0706
-2.0435	-1.4961
-0.2428	-1.3760
0.3091	0.3845

S(0)	sd(1)	S(1)	S*(1)	sd(2)	S(2)	S*(2)	Min(S*(t))	P
51	0.099	50.792	50.861	0.110	51.023	51.078	50.861	0.216
51	0.099	50.862	50.932	0.106	51.025	51.080	50.932	0.149
51	0.099	51.115	51.185	0.097	51.143	51.198	51.000	0.198
51	0.099	51.122	51.192	0.096	50.966	51.021	51.000	0.021
51	0.099	51.149	51.219	0.096	51.110	51.166	51.000	0.166
51	0.099	50.731	50.800	0.114	51.468	51.523	50.800	0.724
51	0.099	51.104	51.174	0.097	51.129	51.184	51.000	0.184
51	0.099	50.470	50.538	0.131	49.960	50.013	50.013	0.000
51	0.099	50.942	51.012	0.103	50.573	50.627	50.627	0.000
51	0.099	51.088	51.158	0.097	51.194	51.250	51.000	0.250
Average		50.937			50.959			
Discounted average (or Monte Carlo price) =								0.1906

Note: P = max[S*(2)-min(S*(t);t=0,1,2),0]

- **GARCH delta and vega hedging**

Static call option delta (see Duan, 1995):

$$\delta_t = \frac{\partial C(S_t, \sigma_{t+1}^2; T, X, r, \Phi)}{\partial S_t}$$
$$= e^{-r(T-t)} E_t^Q\left(\frac{S_T}{S_t}\chi_{\{S_T \geq X\}}\right)$$

Call option vega:

$$\Lambda_t = \frac{\partial C(S_t, \sigma_{t+1}^2; T, X, r, \Phi)}{\partial \sigma_{t+1}^2}$$
$$= \frac{1}{2} e^{-r(T-t)} E_t^Q\left(S_T \chi_{\{S_T \geq X\}} \sum_{\tau=t+1}^{T} (\frac{\varepsilon_\tau^*}{\sigma_\tau} - 1) G_{t+1,\tau-t-1}\right)$$

where

$$G_{t,k} = G_{t,k-1}\left(\beta_1 + \beta_2(\varepsilon_\tau^* - \theta - \lambda)^2\right) \text{ and } G_0 = 1$$

Dynamic call option delta:

$$\delta_t^* = \delta_t + 2\Lambda_t \frac{\beta_2(\varepsilon_\tau^* - \theta - \lambda)}{S_t}$$

because

$$\begin{aligned}\Delta C(S_t, \sigma_{t+1}^2; T, X, r, \Phi) &\cong \delta_t \Delta S_t + \Lambda_t \Delta\sigma_{t+1}^2 \\ &\cong \left(\delta_t + \Lambda_t \frac{\partial\sigma_{t+1}^2}{\partial S_t}\right)\Delta S_t \\ &= \left(\delta_t + 2\Lambda_t \frac{\beta_2(\varepsilon_\tau^* - \theta - \lambda)}{S_t}\right)\Delta S_t\end{aligned}$$

- **Risk-neutral probabilities**

$$\Pr_t^Q\{S_T \le x\} = E_t^Q\left(\chi_{\{S_T \le x\}}\right)$$

The risk-neutral probability can be easily evaluated using the EMS.

One can thus use the estimated GARCH parameter values to generate a risk-neutral probability distribution. This is a better approach than other schemes, such as Shimko (1993) and Rubinstein (1994), for obtaining the risk-neutral probability distribution.

7. Option pricing under stochastic volatility

- **Data generating system**

Divide $[0,T]$ into nT subintervals of length $s=\frac{1}{n}$. Let $\varepsilon_k, k=1,2,\cdots$ be a sequence of i.i.d. standard normal random variables.

$$\ln S_{ks}^{(n)} - \ln S_{(k-1)s}^{(n)} = (r + \lambda\sigma_{ks}^{(n)} - \frac{1}{2}\sigma_{ks}^{(n)2})s + \sigma_{ks}^{(n)}\varepsilon_k\sqrt{s}$$

$$\begin{aligned}\sigma_{(k+1)s}^{(n)2} - \sigma_{ks}^{(n)2} &= \beta_0 s + \sigma_{ks}^{(n)2}[\beta_1 + \beta_2(1+\theta^2) - 1]s \\ &\quad + \sigma_{ks}^{(n)2}[\beta_2(\varepsilon_k - \theta)^2 - \beta_2(1+\theta^2)]\sqrt{s}\end{aligned}$$

<u>Note that if s = 1, we have the NGARCH(1,1)-mean model.</u>

As n goes to infinity, the approximating model becomes

$$d\ln S_t = (r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_{1,t}$$

$$\begin{aligned}d\sigma_t^2 &= \left(\beta_0 + [\beta_1 + \beta_2(1+\theta^2) - 1]\sigma_t^2\right)dt \\ &\quad - 2\theta\beta_2\sigma_t^2 dW_{1,t} + \sqrt{2}\beta_2\sigma_t^2 dW_{2,t}\end{aligned}$$

where $W_{1,t}$ and $W_{2,t}$ are two independent standard Brownian motions.

- **Locally risk-neutralized pricing system**

Denote the risk-neutralized probability law by Q. Let $\varepsilon_k^* = \varepsilon_k + \lambda\sqrt{s}$, for $k = 1,2,\cdots$. They form a sequence of i.i.d. standard normal random variables with respect to Q. The pricing system becomes

$$\ln S_{ks}^{(n)} - \ln S_{(k-1)s}^{(n)} = (r - \frac{1}{2}\sigma_{ks}^{(n)2})s + \sigma_{ks}^{(n)}\varepsilon_k^*\sqrt{s}$$

$$\sigma_{(k+1)s}^{(n)2} - \sigma_{ks}^{(n)2} = \beta_0 s + \sigma_{ks}^{(n)2}[\beta_1 + \beta_2(1+\theta^2) - 1]s + \sigma_{ks}^{(n)2}[\beta_2(\varepsilon_k^* - \theta - \lambda\sqrt{s})^2 - \beta_2(1+\theta^2)]\sqrt{s}$$

Note that if $s = 1$, we have the NGARCH(1,1) option pricing model.

As n goes to infinity, the approximating model becomes

$$d\ln S_t = (r - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_{1,t}^*$$

$$d\sigma_t^2 = \left(\beta_0 + [\beta_1 + \beta_2(1+\theta^2) - 1 + 2\theta\beta_2\lambda]\sigma_t^2\right)dt - 2\theta\beta_2\sigma_t^2 dW_{1,t}^* + \sqrt{2}\beta_2\sigma_t^2 dW_{2,t}^*$$

where $W_{1,t}^*$ and $W_{2,t}^*$ are two independent standard Brownian motions under Q.

Note that this is the pricing result directly deduced from the GARCH option pricing theory.

- **Numerical performance of the GARCH approximation**

Note: The following two tables are taken from Ritchken and Trevor (1999), "Pricing Options under Generalized GARCH and Stochastic Volatility Processess."

<u>Comparison using Monte Carlo simulations</u>

Table 6
Generalized GARCH Prices Compared with Bivariate Diffusion Prices*

	Option Maturity (Days)					
m	**10**	**25**	**50**	**75**	**100**	**200**
			Generalized GARCH Model			
1	1.303 1.328	2.060 2.100	2.920 2.978	3.584 3.655	4.140 4.222	5.795 5.913
2	1.306 1.331	2.064 2.104	2.922 2.980	3.585 3.656	4.140 4.222	5.795 5.914
4	1.307 1.332	2.066 2.106	2.925 2.982	3.588 3.659	4.144 4.226	5.799 5.918
8	1.308 1.333	2.067 2.107	2.926 2.983	3.590 3.661	4.145 4.227	5.803 5.922
12	1.309 1.334	2.069 2.109	2.928 2.985	3.591 3.662	4.144 4.227	5.801 5.920
24	1.310 1.335	2.070 2.110	2.929 2.987	3.593 3.664	4.146 4.229	5.804 5.922
48	1.311 1.336	2.071 2.111	2.930 2.988	3.594 3.665	4.147 4.229	5.805 5.924
			Bivariate Diffusion Model			
48	1.313 1.338	2.071 2.111	2.932 2.989	3.594 3.665	4.149 4.231	5.804 5.922

* Table 6 shows the 95% confidence intervals for the prices of at-the-money European call options obtained using 100,000 simulations of the generalized GARCH model and its bivariate diffusion limit. The parameters are the same as those for Table 1, except that each day is partitioned into *m* trading periods. (The underlying process is not an exact GARCH process for $m \neq 1$.) As the number of trading periods increase, the generalized GARCH model prices converge to those from the bivariate diffusion model.

Convergence speed of Ritchken and Trevor's lattice algorithm

Table 7
Convergence of Generalized GARCH Option Prices*

Maturity (Days)	Trading Periods per Day (m) 1	2	3	4	5	Diffusion Limit ∞_L , ∞_U
2	0.589	0.617	0.603	0.598	0.595	0.580, 0.591
5	0.909	0.939	0.932	0.933	0.931	0.922, 0.939
10	1.312	1.318	1.315	1.315	1.315	1.313, 1.338
20	1.857	1.859	1.860	1.860	1.860	1.854, 1.890
50	2.942	2.943	2.943	2.942	2.942	2.932, 2.989
100	4.165	4.165	4.165	4.164	4.163	4.149, 4.231
200	5.893	5.893	5.893	5.893	5.893	5.804, 5.922

* Table 7 shows the convergence of at-the-money European call option prices as the number of trading periods per day, m, increases. (The other parameters are the same as for Table 1, with n=1.) Over each trading period, the price can move to one of three values. The last two columns show 95% confidence intervals for the true price based on 100,000 simulations of the limiting bivariate diffusion model with m=48. The table clearly shows that for contracts with maturities greater than 20 days, m=1 will suffice for the lattice.

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