Abitrage Approach to Pricing Derivatives

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Static vs. dynamic spanning

- Assume that there are $N$ possible states at time 1. Every security entitles its holder an $N$-dimensional payoff vector. There are $K$ securities with the $N \times K$ payoff matrix $A$ and current $K \times 1$ price vector $P$.

- If $A$ has a rank $N$, then the market is complete in the sense that any possible payoff structure can be spanned (static) by some portfolio of $K$ securities.

- If $A$’s rank is less than $N$, the market is incomplete. A payoff structure can still be priced by arbitrage as long as it falls inside the static spanning.

- Arrow-Debreau equilibrium refers to a complete market competitive equilibrium in which allocations are efficient. Note that no arbitrage is a necessary condition of market equilibrium.

- The price vector $P$ cannot be arbitrary. To say the least, it cannot permit arbitrage in the sense that any two portfolios with an identical payoff vector must have the same current value.

- If one is allowed to trade between time 0 and 1, the spanning set can be enlarged even though the number of securities remains fixed. In other words, one is more likely to be able to price a payoff structure by arbitrage.
Black-Schole dynamic spanning approach to option valuation

Asset price dynamic

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \]

Its derivative security with payoff function at time \( T \) equal to \( f(S_T; \theta) \) has a time-\( t \) value expressed as \( C(S_t, t; \sigma, \mu, T, r, \theta) \) or \( C_t \) for short.

Consider a dynamically rebalanced portfolio shorting \( \Delta_t \) units of the underlying asset to hedge the derivative security. The hedged portfolio’s value at time \( t \) is

\[ V_t = C_t - \Delta_t S_t. \]

Applying Ito’s lemma gives rise to

\[ dV_t = dC_t - \Delta_t dS_t \]

\[ = \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S_t} dS_t + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} dt - \Delta_t dS_t \]

\[ = \left( \frac{\partial C_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} \right) dt + \left( \frac{\partial C_t}{\partial S_t} - \Delta_t \right) dS_t. \]

Setting \( \Delta_t = \frac{\partial C_t}{\partial S_t} \) yields a locally risk-free hedged portfolio. Excluding arbitrage, it must be true that
\[ \left( \frac{\partial C_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} \right) dt = rV_t dt \]

\[ = r \left( C_t - S_t \frac{\partial C_t}{\partial S_t} \right) dt \]

or (the Black-Scholes PDE)

\[ \frac{\partial C_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} + rS_t \frac{\partial C_t}{\partial S_t} - rC_t = 0 \]

The solution to this PDE depends on the terminal condition: \( f(S_T; \theta) \). It can be solved using separation of variables, Green’s function or Fourier/Laplace transformation technique.

**A probabilistic way of solving the generalized Black-Scholes PDE**

When both \( \mu_t \) and \( \sigma_t \) are functions of \( S_t \), the Black-Scholes PDE applies.

\[ \frac{\partial C_t}{\partial t} + \frac{1}{2} \sigma_t^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} + rS_t \frac{\partial C_t}{\partial S_t} - rC_t = 0 \]

The solution to the generalized PDE can be obtained by directly applying the backward equation for the Kac functional; that is the following conditional expectation satisfying the Black-Scholes PDE:
\[ C_t = E\left\{ e^{-r(T-t)} f(S_T; \theta) \mid S_t \right\} \]

with respect to the following artificial diffusion system:

\[ \frac{dS_t}{S_t} = rdt + \sigma_t dW_t^* \]

This probabilistic solution suggests a new perspective of risk-neutral valuation

**Martingale pricing theory**

The Kac functional result suggests that \( e^{-rT}S_t \) is a martingale with respect to the law \( Q \) which \( W_t^* \) is a standard Brownian motion. Note that \( C_T = f(S_T; \theta) \). The same martingale result is thus true for derivatives as well.

Alternatively, one can show this by the Kunita-Watanabe martingale representation theorem (see Harrison and Kreps (1979), *Journal of Economic Theory*).