Financial Time Series

Topic 7: ARCH Related Models

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OUTLINE

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Stock Volatility

- Volatility: the conditional variance of the underlying asset returns

- Black-Scholes option pricing formula states that the price of a European call option is

\[ c_t = P_t \Phi(x) - K r^{-\ell} \Phi(x - \sigma_t \sqrt{\ell}), \]

where

\[ x = \frac{\ln(P_t/K r^{-\ell})}{\sigma_t \sqrt{\ell}} + \frac{1}{2} \sigma_t \sqrt{\ell}. \]

- \( K \): striking price
- \( \ell \): the time to expiration
- \( P_t \): current price of the underlying stock
- \( r \): risk-free interest rate
- \( \sigma_t \): the conditional standard deviation of the log return of the specified stock

- The conditional variance of a stock return plays an important role in the pricing formula.

- How do we model the evolution of stock volatility?
• Conditional heteroscedastic models
  – Shocks of asset returns are NOT serially correlated, but dependent.
  – See ACF of squared and absolute returns of some stocks.

• Univariate volatility models:
  – G(eneralized)ARCH: Bollerslev (1986, J. of *Econometrics*)
    Modeling asymmetry in volatility
  – Stochastic volatility model: Melino and Turnbull (1990, J. of *Econometrics*)

• Stock volatility is not directly observable.
  – The daily volatility is not directly observable from the daily returns.
  – If intra-daily prices are available, one can discuss the daily volatility.
Empirical Properties of Returns

- Empirical research on returns distributions has been ongoing since the early 1960s.
  - Daily returns of the market indexes and individual stocks tend to have high excess kurtoses.
  - Monthly returns have higher standard deviations than daily returns.
  - The skewness is not a serious problem for both daily and monthly returns.

- Volatility process: Study the evolution of conditional variances of the return over time.
  - Figures 2 and 3: The variabilities of returns vary over time and appear in clusters.
  - Extremes of a return series: large positive or negative returns
  - Volatility clusters: high for certain time periods and low for other periods
  - Volatility evolves over time in a continuous manner.
− Volatility varies within some fixed range. Volatility is stationary.
− Volatility seems to react differently to a big positive return and a big negative return.

EGARCH: capture the asymmetry in volatility induced by big positive and negative asset returns.

Study on Volatility

• \( x_t \): the log return of a stock at time index \( t \)

• \( x_t \) is serially uncorrelated or with minor lower lag serial correlations, but it is dependent.

• Volatility models attempt to capture such dependence in the return series.

• Let \( \mathcal{F}_{t-1} \) be the information available at time \( t - 1 \). Consider conditional mean and variance

\[
\begin{align*}
\mu_t &= E(x_t|\mathcal{F}_{t-1}) = g(\mathcal{F}_{t-1}), \\
\sigma_t^2 &= Var(x_t|\mathcal{F}_{t-1}) = h(\mathcal{F}_{t-1}),
\end{align*}
\]

where \( g(\cdot) \) and \( h(\cdot) \) are well-defined functions with \( h(\cdot) > 0 \).
• It is common to assume that $\mu_t = \mu$.

• For a linear series, $g(\cdot)$ is a linear function of $\mathcal{F}_{t-1}$ and $h(\cdot) = \sigma_a^2$.
  
  In statistical literature, they focus on $g(\cdot)$. Model $x_t$ as a stationary $ARMA(p, q)$.
  
  $$
  x_t = \mu_t + a_t,
  $$
  $$
  \mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i x_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i}.
  $$

• $\sigma_t^2$ is $\text{Var}(a_t | \mathcal{F}_{t-1})$. 


Martingales and Random Walks

• A martingale is a stochastic process \( \{x_t\} \) with the following properties:
  
  - \( E(|x_t|) < \infty \) for each \( t \);
  
  - \( E(x_t|\mathcal{F}_s) = x_s \), whenever \( s \leq t \).
  
  \( \mathcal{F}_s \): the \( \sigma \)-algebra comprising events determined by observations over the interval \([0, t]\)
  
  \( \mathcal{F}_s \subset \mathcal{F}_t \) when \( s \leq t \)
  
  - Right continuous: \( \mathcal{F}_t = \cap_{s \geq t} \mathcal{F}_s \)

• \( \mathcal{F}_t = \sigma(x_s; s \leq t) \): the past history of \( \{x_t\}_0^t \) itself

\[
E(x_t - x_s|\mathcal{F}_s) = 0, \quad s \leq t. \tag{1}
\]

• (1) can be written equivalently as

\[
x_t = x_{t-1} + a_t
\]

where \( a_t \) is the martingale increment or martingale difference.

Is it a random walk model?

  - The martingale rules out any dependence of the conditional expectation of \( x_t - x_{t-1} \) on the information available at \( t \).
– The random walk rules out not only any dependence of the conditional expectation of \( x_t - x_{t-1} \) on the information available at \( t \) also dependence involving the higher conditional moments of \( x_t - x_{t-1} \).

– Financial series are known to go through protracted quiet periods and also protracted periods of turbulence. This type of behavior could be modelled by a process in which successive conditional variances of \( x_t - x_{t-1} \) (but not successive levels) are positive correlated. Such a specification would be consistent with a martingale, but not with the more restrictive random walk.

- Submartingale: \( E(x_t - x_s | \mathcal{F}_s) \geq 0, \ s \leq t \).
- Supermartingale: \( E(x_t - x_s | \mathcal{F}_s) \leq 0, \ s \leq t \).

Non-Linearity

- For random walk, 
  \( a_t \) is \( WN(0, \sigma^2) \) (i.e., stationary, uncorrelated, from a fixed distribution)  
  \( a_t \) is \( SWN(0, \sigma^2) \) (i.e., independent too)
• For martingale differences, $a_t$ can be non-stationary.

• Why do we consider the dependence between conditional variances?

• Financial time series often go through protracted quiet periods interspersed with bursts of turbulence.

• Use non-linear stochastic processes to model such volatility.

• Suppose $x_t$ is generated by the process $\Delta x_t = \eta_t$ with

$$\eta_t = a_t + \beta a_{t-1} a_{t-2},$$

where $a_t$ is $SWN(0, \sigma^2)$. 
• Properties of $\eta_t$:

\[ E(a_t) = 0, \]
\[ V(a_t) = \text{constant}, \]
\[ E(\eta_t \eta_{t-k}) = E(a_ta_{t-k} + \beta a_{t-1}a_{t-2}a_{t-k} + \beta a_{t-1}a_{t-k-1}a_{t-k-2} + \beta^2 a_{t-1}a_{t-2}a_{t-k-1}a_{t-k-2}). \]

• For all $k \neq 0$, each of the term in the ACF has zero expectation.

$\eta_t$ behaves like an independent process.

• The conditional expectation is

\[ \hat{\eta}_{t+1} = E(\eta_{t+1}|\eta_t, \eta_{t-1}, \cdots) = \beta a_t a_{t-1}. \]

• $x_t$ is not a martingale because

\[ E(x_{t+1} - x_t|\eta_t, \eta_{t-1}, \cdots) = \hat{\eta}_{t+1} \neq 0. \]
Testing the Random Walk Hypothesis

• Autocorrelation tests:
  – Suppose $w_t = \Delta x_t$ is $SW N(0, \sigma^2)$.
  – The sample autocorrelations (standardized by $\sqrt{T}$) calculated from the realization $\{w_t\}_1^T$ will be $N(0, 1)$.
  – Reject the hypothesis if, for example, $\sqrt{T}|r_1| > 1.96$.
  – Portmanteau tests: $Q^*(K)$ and $Q(K)$.
  – Those tests rely on the assumption that the random walk innovation is strict white noise.
    Refer to page 126 for further discussion.

• Calendar effects:
  – Consider autocorrelations associated with specific timing patterns.
  – January effect: Stock returns in this month are exceptionally large.
  – Weekend effect: Monday mean returns are negative rather than positive as for all other weekdays.
– Holiday effect: a much larger mean return for the day before holidays
– Turn-of-the-month effect: the four-day return around the turn of a month is greater than the average total monthly return
– Intramonth effect: the return over the first half of a month is significantly larger than the return over the second half
Stochastic Volatility

- Allow the variance (or the conditional variance) of the process to change either at certain discrete points in time or continuously.
- A stationary process must have a constant variance, certain conditional variances can change.
- For a non-linear stationary process $x_t$, the variance $Var(x_t)$ is a constant for all $t$, but the conditional variance $Var(x_t | x_{t-1}, x_{t-2}, \ldots)$ can change from period to period.
- Non-stationary variance or variance dependent on past observations and additional variables
- The models are non-linear, have high kurtosis, and positive autocorrelation between squared returns.

Stochastic volatility (SV) models

- $\{x_t\}_1^t$: the product process

\[
x_t = \mu + \sigma_t U_t
\]  

(2)
where 
\[ E(U_t) = 0 \] and \[ Var(U_t) = 1 \] for all \( t \),
\[ Var(x_t | \sigma_t) = \sigma_t^2, \] and
\( \sigma_t \) is a positive random variable.

- \( E(x_t) = \mu, \)
  \[ E(x_t - \mu)^2 = E(\sigma_t^2 U_t^2) = E(\sigma_t^2), \]
and autocovariance
  \[ E(x_t - \mu)(x_{t-k} - \mu) = E(\sigma_t \sigma_{t-k} U_t U_{t-k}) = E(\sigma_t \sigma_{t-k} U_t) E(U_{t-k}) = 0. \]

- Typically \( U_t = (x_t - \mu)/\sigma_t \) is assumed to be normal and independent of \( \sigma_t \).

- (2) is motivated by the discrete time approximation to the stochastic differential equation
  \[ \frac{dP}{P} = d(\log(P)) = \mu dt + \sigma dW \]
where \( x_t = \Delta \log(P_t) \) and \( W(t) \) is standard Brownian motion.
This is the usual diffusion process used to price financial assets in theoretical models of finance.
• In the world of time series analysis, write the above sdf by setting $dt = 1$. We then have

$$\log(P_{t+1}) - \log(P_t) = \mu + \sigma(W_{t+1} - W_t).$$

• Although $x_t$ is a white noise, the squared and absolute deviation, $S_t = (x_t - \mu)^2$ and $M_t = |x_t - \mu|$, can be autocorrelated.

$$Cov(S_t, S_{t-k}) = E(\sigma^2_t \sigma^2_{t-k}) E(U^2_t U^2_{t-k}) - (E(\sigma^2_t))^2$$

$$= E(\sigma^2_t \sigma^2_{t-k}) - (E(\sigma^2_t))^2.$$  

• Fact: Almost all sample paths $W$ of Brownian motion are of unbounded variation. They are not differentiable.
How do we model $\sigma_t$?

- The distribution of $\sigma_t$ is skewed to the right. Consider a log-normal distribution.

- Define

$$h_t = \log(\sigma_t^2) = \gamma_0 + \gamma_1 h_{t-1} + \eta_t \quad (3)$$

where $\eta_t \sim NID(0, \sigma_\eta^2)$ and is independent of $U_t$.

$h_t$ represents the random and uneven flow of new information into financial market.

- $x_t = \mu + U_t \exp(h_t/2)$, where $U_t$ is always stationary.

$x_t$ will be stationary if and only if $h_t$ is.

Or, $|\gamma_1| < 1$.

- Moments of $x_t$ or $S_t$:

For even $r$,

$$E(x_t - \mu)^r = E(U_t^r) E(\exp(r h_t/2))$$

$$= \frac{r!}{2^r/2^r/2!} \exp \left( \frac{r}{2} \mu_h + \frac{r}{2} \sigma_h^2 \right)$$

where $\mu_h = E(h_t) = \gamma_0/(1 - \gamma_1)$ and $\sigma_h^2 = V(h_t) = \sigma_\eta^2/(1 - \gamma_1^2)$.

All odd moments are zero.
• kurtosis:
\[
\frac{E(x_t - \mu)^4}{[E(x_t - \mu)^2]^2} = 3\exp(\sigma_h^2) > 3
\]
The process has fatter tails than a normal distribution.

• autocorrelation: Refer to page 129.

• Taking logarithms of (2) yields
\[
\log(S_t) = h_t + \log(U_t^2)
\]
\[
= \mu_h + \frac{\eta_t}{1 - \gamma_1 B} + \log(U_t^2)
\]
\[
\log(S_t) \sim ARMA(1, 1)
\]
with non-normal innovations.

• The main difficulty with using SV models is that they are rather difficult to estimate.
ARCH Processes

- In (3), $\sigma_t$ was dependent upon the information set $\{\eta_t, \sigma_{t-1}, \sigma_{t-2}, \ldots\}$.

- Now, consider the case that $\sigma_t$ are a function of past values of $x_t$,

$$\sigma_t^2 = h(x_{t-1}, x_{t-2}, \ldots).$$

- ARCH(1) process: Engle (1982)

- First-order autoregressive conditional heteroskedastic process:

  Write $\epsilon_t$ as $\sigma_t U_t$ where $\{U_t\}$ is a sequence of iid r.v. with mean 0 and variance 1.

  $$\sigma_t^2 = h(x_{t-1}) = \alpha_0 + \alpha_1 \epsilon_{t-1}^2, \quad (4)$$

  where $\alpha_0, \alpha_1 > 0$.

  - The (mean-corrected) asset return $x_t$ is serially uncorrelated but dependent.

  - The dependence of $x_t$ can be described by a simple quadratic function.

  - Large deviations of $x_{t-1}$ from the mean $\mu$ then cause a large variance for the next day.
Large returns tend to be followed by another large return

- Distribution of $U_t$: standard normal, standardized Student-$t$, or generalized error distribution.
- When $U_t \sim NID(0, 1)$ and independent of $\sigma_t$,
  \[ x_t = \mu + U_t \sigma_t \]
  is white noise and conditionally normal, i.e.
  \[ x_t|x_{t-1}, x_{t-2}, \ldots \sim NID(\mu, \sigma_t^2) \]
  so that
  \[ Var(x_t|x_{t-1}) = \alpha_0 + \alpha_1(x_{t-1} - \mu)^2. \]
  - $E(x_t) = E[E(x_t|\mathcal{F}_{t-1})] = \mu$
  - Unconditional variance:
    \[ Var(x_t) = E[E(U_t^2 \sigma_t^2|\mathcal{F}_{t-1})] \]
    \[ = \alpha_0 + \alpha_1 E(x_{t-1} - \mu)^2. \]
  - Because $x_t$ is a stationary process, we have $Var(x_t) = \alpha_0/(1 - \alpha_1)$ if $\alpha_1 < 1$.
  - It possesses constant variance yet changing conditional variance.
– When $0 < \alpha_1^2 < 1/3$, the fourth moment is finite.
\begin{align*}
E(U_t)^2 &= 3[Var(U_t)]^2 \times \frac{1 - \alpha_1^2}{1 - 3\alpha_1^2}.
\end{align*}

– The fourth moment of $U_t$ is greater than that of a normal random variable when $\alpha_1 \neq 0$.
This implies that the $U_t$ process is heavy-tailed and it is capable of producing clusters of outliers.

• Using (4), the series of $S_t = (x_t - \mu)^2$ satisfy
\begin{align*}
E(S_t | S_{t-1}) &= \alpha_0 + \alpha_1 S_{t-1}.
\end{align*}
a stationary AR(1) process

• ARCH($q$) process:
\begin{align*}
\sigma_t &= h(x_{t-1}, \ldots, x_{t-q}) \\
&= \left( \alpha_0 + \sum_{i=1}^{q} \alpha_i (x_{t-i} - \mu)^2 \right)^{1/2},
\end{align*}
where $\alpha_0$ and $\alpha_i \geq 0$, $1 \leq i \leq q$.
$S_t$: an AR($q$) process

• The process is weakly stationary if all the roots of the characteristic equation associ-
ated with the ARCH parameters, $\alpha(B)$, lie outside the unit circle, i.e., if $\sum_{i=1}^{q} \alpha_i < 1$.

• Unconditional variance:
  \[
  \alpha_0 / (1 - \sum_{i=1}^{q} \alpha_i)
  \]

• Conditional variance:
  \[
  \sigma_i^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2
  \]
  or
  \[
  \epsilon_i^2 = \alpha_0 + \alpha(B) \epsilon_{t-i}^2 + v_t.
  \]

• Weakness of ARCH models:
  - This model treats positive and negative returns in the same manner, because it depends on the square of the previous returns.
    In practice, it is well-known that for financial time series the prices respond differently to positive and negative returns.
  - The ARCH model is rather restrictive. For the ARCH(1) model of $\alpha_1^2$ must be between 0 and $1/3$. For higher-order ARCH models, the constraint is even stronger.
ARCH models often over-predict the volatility, because they respond slowly to isolated large stocks to the return series.
Building ARCH Models

• Step 1: Remove the linear dependence of the return series and test for ARCH effects.
  – Mean Effect: Build an ARIMA model for the observed time series to remove any serial correlations in the data.
  – For most asset return series, this step amounts to remove the sample mean from the data if the sample mean is significantly different from zero.
  – Define $\epsilon_t = x_t - \mu_t$.
  – Examine the squared series $\epsilon_t^2$ to check for conditional heteroscedasticity.

• Step 2: Order determination
  If conditional heteroscedasticity is detected, we use the PACF of $\epsilon_t^2$ to determine the ARCH order.

• Step 3: Estimation
  – Conditional MLE
  – Software: S-plus, RATS

• Step 4: The fitted ARCH model is carefully examined and refined if necessary.
skewness, kurtosis, standardized residuals, and etc

Likelihood Function and ARCH Estimation

• Note that

\[ \sigma_t = \left( \alpha_0 + \sum_{i=1}^{q} \alpha_i (x_{t-i} - \mu)^2 \right)^{1/2}. \]

\( \sigma_t \) is a function of \( x_{t-i} - \mu \) (1 \( \leq i \leq q \)) and \( q + 1 \) parameters \( \alpha_i \) (0 \( \leq i \leq q \)).

• Denote by \( \omega \) the set of parameters \( \mu, \alpha_0, \alpha_1, \ldots, \alpha_q \).

• The likelihood function for \( T \) observed returns is

\[
L(x_1, x_2, \ldots, x_T | \omega) = f(x_1 | \omega) f(x_2 | I_1, \omega) \cdots f(x_T | I_{T-1}, \omega).
\]

Here \( f(x_t | I_{t-1}, \omega) \) denotes the conditional density of \( x_t \) given the previous observations \( I_{t-1} = \{x_1, x_2, \ldots, x_{t-1}\} \) and the parameter vector \( \omega \).

• Under the normality assumption, for \( t > q \),

\[
f(x_t | I_{t-1}, \omega) = f(x_t | \sigma_t) = (\sqrt{2\pi}\sigma_t)^{-1} \exp \left[ -\frac{1}{2}(x_t - \mu)^2 / \sigma_t^2 \right].
\]
Or, the likelihood function of is

$$\frac{1}{\sqrt{2\pi \sigma^2_t}} \exp \left[ - \frac{(x_t - \mu)^2}{2\sigma^2_t} \right] \times f(x_1, \ldots, x_q | \omega).$$

- The conditional maximum likelihood estimate $\omega$ for observations $q + 1$ to $T$, maximizes

$$L_q(\omega) = \prod_{t=q+1}^{n} f(x_t | I_{t-1}, \omega).$$

The log likelihood function becomes

$$-\sum_{t=q+1}^{T} \left[ \frac{1}{2} \ln(\sigma^2_t) + \frac{1}{2} \frac{a^2_t}{\sigma^2_t} \right],$$

where $\sigma^2_t = \alpha_0^2 + \alpha_1 a^2_{t-1} + \cdots + \alpha_q a^2_{t-m}$ can be evaluated recursively.
The GARCH Model

- The ARCH model often requires many parameters to adequately described the evolution of volatility of an asset return. For the monthly return series of S&P 500 index, an ARCH(9) model is needed for the volatility series.

- For a log return series $x_t$, the conditional mean $\mu_t$ can be adequately described by an ARMA model. Let $\epsilon_t = x_t - \mu_t$ be the mean-corrected log return.

- Generalized ARCH (GARCH($p,q$)) process: Bollerslev (1986, 1988);

\[
\begin{align*}
\epsilon_t &= \sigma_t U_t, \\
\sigma_t^2 &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2,
\end{align*}
\]

where $\{U_t\}$ is a sequence of iid random variables with mean 0 and variance 1, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{\max(p,q)}(\alpha_i + \beta_i) < 1$.

If $p = 0$, it reduces to a pure ARCH($q$) model.

- Let $\eta_t = \epsilon_t^2 - \sigma_t^2$. We get the following equiv-
alent form:

\[ \epsilon_t^2 = \alpha_0 + \sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i) \epsilon_{t-i}^2 + \eta_t - \sum_{j=1}^{q} \beta_j \eta_{t-j}. \]  

(5)

It is an ARMA form for the squared series \( \epsilon_t^2 \).

A GARCH model can be regarded as an application of the ARMA idea to the squared series \( \epsilon_t^2 \).

- Using the unconditional mean of an ARMA model, we have

\[
E(\epsilon_t^2) = \frac{\alpha_0}{\sum_{i=1}^{\max(p,q)}(\alpha_i + \beta_i)}
\]

provided that the denominator of the above fraction is positive.

- The process is weakly stationary if and only if the roots of \( \alpha(B) + \beta(B) \) lie outside the unit circle, i.e., \( \alpha(1) + \beta(1) < 1 \).

- A popular model for financial time series: GARCH(1, 1) process

\[
\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2.
\]
To be well-defined, $0 \leq \alpha_1, \beta_1 \leq 1$ and $\alpha_1 + \beta_1 < 1$.

- Volatility clustering: A large $\epsilon_{t-1}^2$ or $\sigma_{t-1}^2$ gives rise to a large $\sigma_t^2$.

- Heavy tail: If $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$, then
  \[
  \frac{E(\epsilon_t^4)}{[E(\epsilon_t^2)]^2} = \frac{3[1 - (\alpha_1 + \beta)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3.
  \]

- $\epsilon_t$ and $\sigma_t^2$ are strictly stationary if and only if
  \[
  E(\log(\beta_1 + \alpha_1 U_t^2)) < 0.
  \]
Generalized ARCH

- I(ntergrated)GARCH\((p,q)\):
  
  - If the AR polynomial of the GARCH representation has a unit root, we then have an IGARCH mode.
  
  - IGARCH models are unit-root GARCH models.
  
  - The key feature of IGARCH models is that the impact of past squared shocks \(\eta_{t-i} \ (i > 0)\) on \(\epsilon_t^2\) is persistent.

- \(\alpha(1) + \beta(1) = 1\)
  
Here I refers to integrated. See page 134.

- Consider IGARCH\((1,1)\) model.

\[
\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1)\epsilon_{t-1}^2,
\]

where \(0 < \beta_1 < 1\).

- E(xponential)GARCH model:

  - Allow for asymmetric effects between positive and negative asset returns.

  - Nelson (1991)

\[
\log \sigma_t^2 = \alpha_0 + \alpha_1 f(\epsilon_{t-1}/\sigma_{t-1}) + \beta_1 \log \sigma_{t-1}^2
\]
where
\[
f(\frac{\epsilon_{t-1}}{\sigma_{t-1}}) = \theta_1 \frac{\epsilon_{t-1}}{\sigma_{t-1}} + \\
\left( \frac{\epsilon_{t-1}}{\sigma_{t-1}} - E\left|\frac{\epsilon_{t-1}}{\sigma_{t-1}}\right| \right).
\]

- The asymmetry allows volatility to respond more rapidly to falls in a market than to corresponding rises. See page 137.

- Long memory volatility processes: The FI-GARCH model.
  - FI refers to fractionally integrated.
  - See (4.10) in page 139.