Financial Time Series

Topic 1: Asset Returns

Hung Chen
Department of Mathematics
National Taiwan University
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OUTLINE

1. Introduction
2. Definitions
3. Distribution of Returns
4. Empirical Properties of Returns

Introduction

- Provide some basic knowledge of financial time series and to introduce some statistical tools useful for analyzing these series.
- Basic concepts of asset returns.
- Linear time series analysis
  - stationarity
  - autocorrelation function
  - conditional heteroscedasticity
  - long memory series
- Nonlinear time series
- Nonlinearity of financial time series

Asset Returns

- P_t: The price of an asset at time index t.
- One-period simple return:
  From date $t-1$ to date t,
  $$R_t = \frac{P_t}{1-P_t} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}.$$
- Multiperiod simple return:
  From date $t-k$ to date t,
  $$1 + R_t(k) = \frac{P_t}{P_{t-k}} = \prod_{j=0}^{k-1}(1 + R_{t-j}).$$
  If the asset was held for k years, the annualized return is defined as
  $$\text{Annualized}[R_t(k)]$$
  $$= \left[\prod_{j=0}^{k-1}(1 + R_{t-j})\right]^{1/k} - 1 \approx k^{-1}\sum_{j=0}^{k-1} R_{t-j}.$$
  Care must be exercised in using the approximation.
- Continuously compounded return:
  $$r_t = \ln(1 + R_t)$$
  For multiperiod returns, we have
  $$r_t(k) = r_t + r_{t-1} + \cdots + r_{t-k+1}.$$

- Capital Asset Pricing Model (CAPM): Sharpe (1964, *J. of Finance*, 425-442)
  Consider the joint distribution of N return at a single time index t. Or, study the distribution of $\{R_{1t}, \cdots, R_{Nt}\}$.

- In this course, we emphasize on the dynamic structure of individual asset returns.
  Study the distribution of $\{R_{i1}, \cdots, R_{iT}\}$.

- How do we describe the joint distribution of $\{R_{it}\}_{t=1}^{T}$?

  – It is useful to partition the joint distribution $F(R_{i1}, \cdots, R_{iT}; \theta)$ as

  $$F(R_{i1})F(R_{i2}|R_{i1}) \cdots F(R_{iT}|R_{i,T-1}, \cdots, R_{i1})$$
  $$= F(R_{i1}) \prod_{t=2}^{T} F(R_{it}|R_{i,t-1}, \cdots, R_{i1}),$$

  where θ is a vector of parameters.

  – This partition highlights the temporal dependence of the simple return R_{it} of asset i. The main issue is how the conditional distribution evolves over time.

- One version of the random-walk hypothesis is that

$$F(R_{it}|R_{i,t-1},\cdots,R_{i1}) = F(R_{it}).$$

This means that returns are temporally independent and are not predictable.

- For index returns or lower-frequency returns, we usually treat returns as continuous random variables.
  We use density instead of distribution.
  Write $f(R_{i1},\cdots,R_{iT};\theta)$ as

$$f(R_{i1};\theta)\prod_{t=2}^{T} f(R_{it}|R_{i,t-1},\cdots,R_{i1};\theta).$$

- For high-frequency asset returns, discreteness becomes an issue. For example, stock prices change in multiplies of a tick size in the New York Stock Exchange. The tick size was one eights of a dollar before July 1997 and is one sixteenths of a dollar now.

Marginal Distributions of Asset Returns

- When asset returns have weak empirical serial correlations, their marginal distributions are close to their conditional distributions. Moreover, it is easier to estimate marginal distributions than conditional distributions using past returns.
- Normal distributions:
  A traditional assumption is that the simple returns $\{R_{it}\}_{t=1}^{T}$ are independent and identically distributed as normal with fixed mean and variance.
  It encounters several difficulties:
  - The lower bound of a simple return is -1.
  - If R_{it} is normally distributed, the multiperiod simple return $R_{it}(k)$ is not normally distributed.
  - The normality assumption is not supported by many empirical asset returns, which tend to have excess kurtosis.
- The skewness and kurtosis of a random vari-

able X are defined as

$$S(X) = E(X - \mu_X)^3/\sigma_X^3,$$
$$K(X) = E(X - \mu_X)^4/\sigma_X^4.$$

– The quantity $K(X) - 3$ is called the *excess kurtosis* because $K(X) = 3$ when X is normally distributed.

– A distribution with positive excess kurtosis is said to have heavy tails, implying that the distribution puts more mass on the tails of its support than a distribution does.
In practice, it means that a random sample from such a distribution tends to contain more extreme values.

– Let $\{x_1, \ldots, x_T\}$ be a sample of X with T observations. The sample skewness is

$$\hat{S}(X) = \frac{1}{T\hat{\sigma}_X^3} \sum_{t=1}^{T} (x_t - \hat{\mu}_X)^3,$$

where

$$\hat{\mu}_X = T^{-1} \sum_{t=1}^{T} x_t,$$
$$\hat{\sigma}_X^2 = T^{-1} \sum_{t=1}^{T} (x_t - \hat{\mu}_X)^2.$$

The sample kurtosis is

$$\hat{K}(X) = \frac{1}{T\hat{\sigma}_X^4} \sum_{t=1}^{T} (x_t - \hat{\mu}_X)^4.$$

– Under normality assumption, $\hat{S}(X)$ and $\hat{K}(X)$ are distributed asymptotically as normal with zero mean and variances $6/T$ and $24/T$, respectively.

- Lognormal distribution:
  Another commonly used assumption is that the log returns r_t of an asset is independent and identically distributed as normal with mean μ and variance σ^2.

  – The simple returns are then iid lognormal random variables with mean and variance given by

$$\begin{aligned} E(R_t) &= \exp(\mu + \sigma^2/2), \\ Var(R_t) &= \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]. \end{aligned}$$

  – Let m_1 and m_2 be the mean and variance of the simple return R_t. Then the mean and variance of the corresponding log return r_t are

$$\begin{aligned} E(r_t) &= \ln\left[\frac{m_1+1}{\sqrt{1+m_2/(1+m_1)^2}}\right], \\ Var(r_t) &= \ln\left[1+m_2/(1+m_1)^2\right]. \end{aligned}$$

– Because sum of a finite umber of iid normal random variables is normal, $r_t(k)$ is also normally distributed under the normal assumption for $\{r_t\}$.

– There is no lower bound for r_t and the lower bound for R_t is satisfied using $1 + R_t = \exp(r_t)$.

– But the lognormal assumption is not consistent with all the properties of historical stock returns.

- Stable distribution:
  It is capable of capturing excess kurtosis shown by historical stock returns.

  – Cauchy distribution is a stable distribution. The density function of Cauchy is

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

It is symmetric against its mean 0, but has infinite variance.

- Nonnormal stable distributions do not have a finite variance, which is in conflict with most finance theories.
- The stable distribution is stable under addition.

- Scale-Mixture of Normal Distribution:
  It assumes that the log return r_t is normally distributed $N(\mu, \sigma^2)$.
  - But σ^2 is a random variable that follows a positive distribution.
  - As an example,σ^2 will take on two possible values σ_1^2 and σ_2^2 with probability $1 - \alpha$ and α where $0 \leq \alpha \leq 1$. Then $r_t \sim (1 - \alpha)N(\mu, \sigma_1^2) + \alpha N(\mu, \sigma_2^2)$.
  - σ_2^2 is much larger than σ_1^2. When $\alpha = 0.05$, it says that majority of the returns follow a simple normal distribution but also capture the property of excess kurtosis.

Empirical Properties of Returns

- Figure 1 shows the probability density functions of finite-mixture of normal, Cauchy, and standard normal distributions.
- Figure 2 shows the time plots of monthly simple returns and log returns of IBM from January 1927 to December 1997.
- Figure 3 shows the same plots for the monthly returns of value-weighted market index.
- Figures 3 and 4 show that the basic patterns of simple and log returns are similar.
- Figure 4 shows the empirical density functions of monthly simple and log returns of IBM.
  - Also shown, by a smooth line, in each graph is the normal probability density evaluated using the sample mean and standard deviation of IBM given in Table 2.
  - The plots indicates that the normality is questionable for monthly IBM returns. The empirical density function has a higher

peak around its mean, but fatter tails than normal.

- Table 2 provides some summary statistics of simple and log returns for selected U.S. market indexes and individual stocks.
  - Daily returns of the market indexes and individual stocks tend to have high excess kurtoses.
  - For monthly series, the returns of market indexes have higher excess kurtoses than individual stocks.
  - The mean of a daily return series is close to zero whereas that of a monthly return series is slightly larger.
  - Monthly returns have higher standard deviations than individual stocks.
  - The skewness is not a serious problem for both daily and monthly returns.
  - The summary statistics show that the difference between simple and log returns are not substantial.

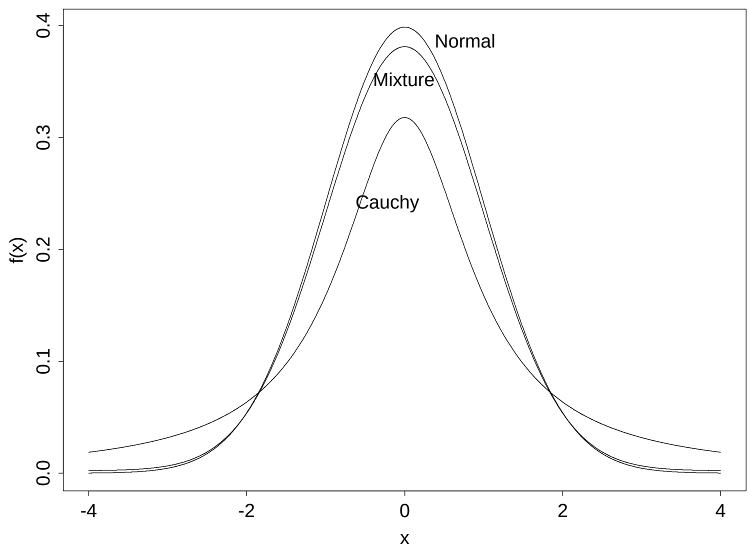

Figure 1: Comparison of Finite-mixture, Stable, and Standard Normal Density Functions.

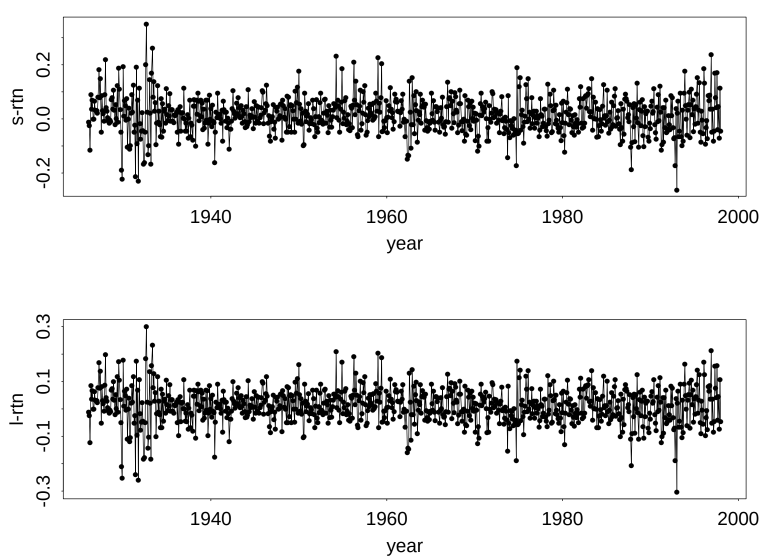

Figure 2: Time plots of monthly returns of IBM from January, 1926 to December, 1997. The upper panel is for simple net returns and the lower panel is for log returns.

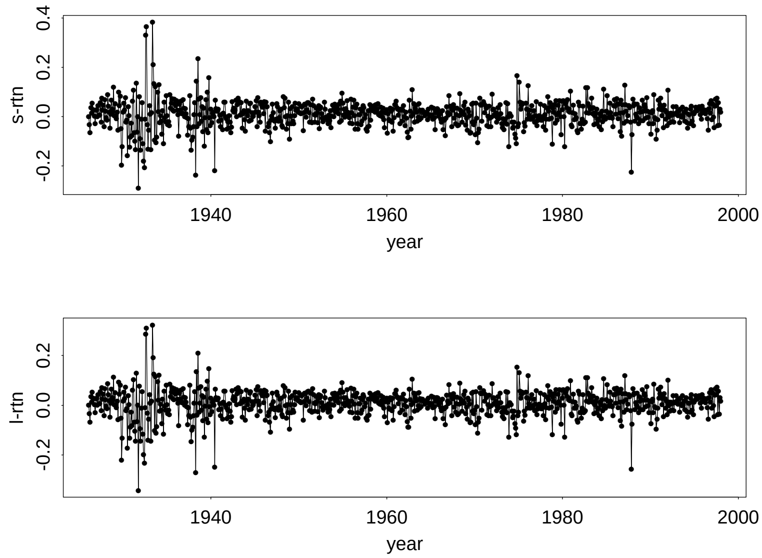

Figure 3: Time plots of monthly returns of the value-weighted index from January, 1926 to December, 1997. The upper panel is for simple net returns and the lower panel is for log returns.

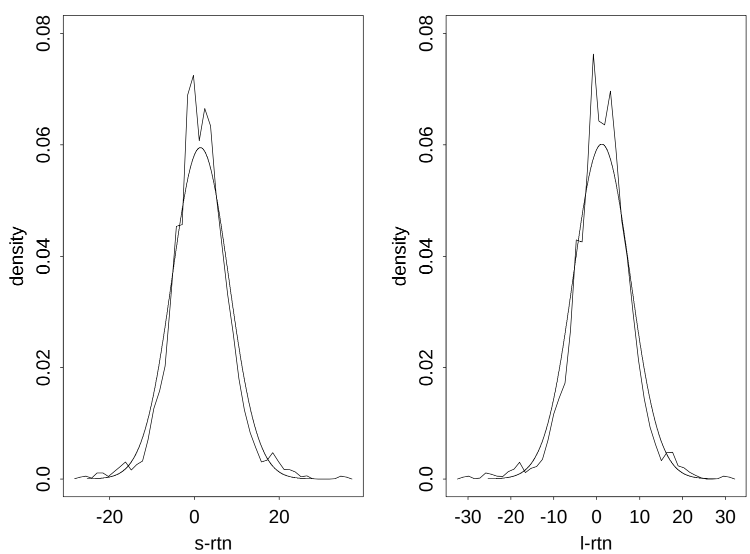

Figure 4: Comparison of empirical and normal densities for the monthly simple and log returns of IBM. The sample period is from January, 1926 to December, 1997. The left plot is for simple returns and the right plot for log returns. The normal density, shown by the smooth line, uses the sample mean and standard deviation given in Table 2.

Table 2: Summary Statistics for Daily and Monthly Simple and Log Returns of Selected Indexes and Stocks. Returns are in percent and the sample period ends at December 31, 1997. The summary statistics are defined in equations (10)-(13).

Security	Start	size	Mean	Standard Devia.	Skew.	Excess Kurt.	Min.	Max.
			(a) Daily simple returns (in percent)					
Value-weighted index	62/7/3	8938	0.049	0.798	−1.23	30.06	−17.18	8.67
Equal-weighted index	62/7/3	8938	0.083	0.674	−1.09	18.09	−10.48	6.95
Intern. Bus. Machines	62/7/3	8938	0.050	1.479	0.01	11.34	−22.96	12.94
Intel	72/12/15	6329	0.138	2.880	−0.17	6.76	−29.57	26.38
3M	62/7/3	8938	0.051	1.395	−0.55	16.92	−25.98	11.54
Microsoft	86/3/14	2985	0.201	2.422	−0.47	12.08	−30.13	17.97
Citi-Group	86/10/30	2825	0.125	2.124	−0.06	9.16	−21.74	20.75
			(b) Daily log returns (in percent)					
Value-weighted index	62/7/3	8938	0.046	0.803	−1.66	40.06	−18.84	8.31
Equal-weighted index	62/7/3	8938	0.080	0.676	−1.29	19.98	−11.08	6.72
Intern. Bus. Machines	62/7/3	8938	0.039	1.481	−0.33	15.21	−26.09	12.17
Intel	72/12/15	6329	0.096	2.894	−0.59	8.81	−35.06	23.41
3M	62/7/3	8938	0.041	1.403	−1.05	27.03	−30.08	10.92
Microsoft	86/3/14	2985	0.171	2.443	−1.10	19.65	−35.83	16.53
Citi-Group	86/10/30	2825	0.102	2.128	−0.44	10.68	−24.51	18.86
			(c) Monthly simple returns (in percent)					
Value-weighted index	26/1	864	0.99	5.49	0.23	8.13	−29.00	38.28
Equal-weighted index	26/1	864	1.32	7.54	1.65	15.24	−31.23	65.51
Intern. Bus. Machines	26/1	864	1.42	6.70	0.17	1.94	−26.19	35.12
Intel	72/12	300	2.86	12.95	0.59	3.29	−44.87	62.50
3M	46/2	623	1.36	6.46	0.16	0.89	−27.83	25.77
Microsoft	86/4	141	4.26	10.96	0.81	2.32	−24.91	51.55
Citi-Group	86/11	134	2.55	9.17	−0.14	0.47	−26.46	26.08
			(d) Monthly log returns (in percent)					
Value-weighted index	26/1	864	0.83	5.48	−0.53	7.31	−34.25	32.41
Equal-weighted index	26/1	864	1.04	7.24	0.34	8.91	−37.44	50.38
Intern. Bus. Machines	26/1	864	1.19	6.63	−0.22	2.05	−30.37	30.10
Intel	72/12	300	2.03	12.63	−0.32	3.20	−59.54	48.55
3M	46/2	623	1.15	6.39	−0.14	1.32	−32.61	22.92
Microsoft	86/4	141	3.64	10.29	0.29	1.32	−28.64	41.58
Citi-Group	86/11	134	2.11	9.11	−0.50	1.14	−30.73	23.18