



Distance-two labelings of graphs

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Abstract

For given positive integers $j \geq k$, an $L(j, k)$ -labeling of a graph G is a function $f : V(G) \rightarrow \{0, 1, 2, \dots\}$ such that $|f(u) - f(v)| \geq j$ when $d_G(u, v) = 1$ and $|f(u) - f(v)| \geq k$ when $d_G(u, v) = 2$. The $L(j, k)$ -labeling number $\lambda_{j,k}(G)$ of G is defined as the minimum m such that there is an $L(j, k)$ -labeling f of G with $f(V(G)) \subseteq \{0, 1, 2, \dots, m\}$. For a graph G of maximum degree $\Delta \geq 1$ it is the case that $\lambda_{j,k}(G) \geq j + (\Delta - 1)k$. The purpose of this paper is to study the structures of graphs G with maximum degree $\Delta \geq 1$ and $\lambda_{j,k}(G) = j + (\Delta - 1)k$.

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1. Introduction

The problem of vertex labeling with a condition at distance two, proposed by Griggs and Roberts [16], arose from a variation of the channel assignment problem introduced by Hale [10]. Suppose a number of transmitters are given. We must assign a channel to each of the given transmitters such that the interference is avoided. In order to reduce the interference, any two “close” transmitters must receive channels at least k apart, and any two “very close” transmitters must receive channels at least j apart, where $j \geq k$ are two given positive integers. One can construct an interference graph for this problem so that the transmitters are the vertices and there is an edge joining two “very close” transmitters. Two transmitters are defined as “close” if the corresponding vertices are of distance two.

Then, for a given graph G , an $L(j, k)$ -labeling is defined as a function $f : V(G) \rightarrow \{0, 1, 2, \dots\}$ such that $|f(u) - f(v)| \geq j$ when $d_G(u, v) = 1$ and $|f(u) - f(v)| \geq k$ when $d_G(u, v) = 2$, where $d_G(u, v)$, the distance of u and v , is the minimum length of a path between u and v . The $L(j, k)$ -labeling number $\lambda_{j,k}(G)$ of G is the smallest number

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m such that G has an $L(j, k)$ -labeling with no label greater than m . A $\lambda_{j,k}$ -labeling of G is an $L(j, k)$ -labeling using labels not greater than $\lambda_{j,k}(G)$.

The $L(j, k)$ -labeling problem, in particular the $L(2, 1)$ case, has been extensively studied during the past decade ([1–9, 11–20]). Among the results, we are most interested in those on exact values, which are very often for trees. For instance, it was shown in [5] that the $\lambda_{1,1}$ -number of any tree T is equal to its maximum degree Δ , implying that $\lambda_{j,j}(T) = j\Delta$. For $j = 2$ and $k = 1$, Griggs and Yeh [9] proved the following result by means of a first-fit (greedy) algorithm: for any tree T with maximum vertex degree $\Delta \geq 1$,

$$\Delta + 1 \leq \lambda_{2,1}(T) \leq \Delta + 2.$$

This result gives rise to the question of classifying those trees with $\lambda_{2,1}(T) = \Delta + 1$ or alternatively $\Delta + 2$. Although Griggs and Yeh conjectured that the classification problem is NP-complete, Chang and Kuo [2] presented a polynomial-time classification algorithm.

The same procedure was also carried out for the $L(j, 1)$ -labelings. It was proved by Chang et al. [1] that for any tree of maximum degree Δ ,

$$\Delta + j - 1 \leq \lambda_{j,1}(T) \leq \min\{\Delta + 2j - 2, 2\Delta + j - 2\}.$$

Moreover, the lower bound and the upper bounds are both attainable. They also gave a polynomial-time algorithm for determining $\lambda_{j,1}(T)$ of a tree T .

We now consider the lower bound on the above two cases for general graphs G and general positive integers $j \geq k$.

Proposition 1. *For any positive integers $j \geq k$ and any graph G of maximum degree $\Delta \geq 1$, we have $\lambda_{j,k}(G) \geq j + (\Delta - 1)k$. Moreover, if $j > k$ and the equality holds, then for any $\lambda_{j,k}$ -labeling of G each major vertex (i.e. a vertex of degree Δ) must be labeled 0 (or $j + (\Delta - 1)k$) and its neighbors must be labeled $j + ik$ (or ik) for $i = 0, 1, \dots, \Delta - 1$.*

We call a graph G of maximum degree $\Delta \geq 1$ with $\lambda_{j,k}(G) = j + (\Delta - 1)k$ a $\lambda_{j,k}$ -minimal graph. It is clear that $K_{1,\Delta}$ is $\lambda_{j,k}$ -minimal for all $\Delta \geq 1$ and $j \geq k$; see [5]. While determining $\lambda_{j,k}(G)$ for a graph is in general not an easy job, the aim of this paper is to study the structures of $\lambda_{j,k}$ -minimal graphs.

2. $\lambda_{j,k}$ -minimal graphs

We first study the structures of $\lambda_{j,k}$ -minimal graphs.

Notice that whether a graph is $\lambda_{j,k}$ -minimal or not depends on the values of j and k . As an example, for $\Delta \geq 1$ consider the *double star* D_Δ which is obtained from two stars $K_{1,\Delta}$ by identifying a leaf of one star with a leaf of the other star. Fig. 1 shows the double star D_6 .

By Proposition 1, if D_Δ is $\lambda_{j,k}$ -minimal then one of the two major vertices is labeled by 0 and its neighbors by $j, j+k, \dots, j+(\Delta-1)k$, and the other is labeled by $j+(\Delta-1)k$ and its neighbors by $0, k, \dots, (\Delta-1)k$. Hence, the only common neighbor of the two major vertices is labeled by $j+i'k = i''k$ for some $0 \leq i', i'' \leq \Delta - 1$. This is possible if and only if $j = ik$ for some integer $1 \leq i \leq \Delta - 1$. In conclusion, D_Δ is $\lambda_{j,k}$ -minimal if and only if $j = ik$ for some integer $1 \leq i \leq \Delta - 1$.

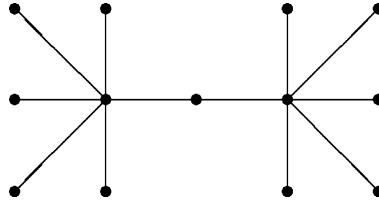


Fig. 1. The double star D_6 .

For some cases, $\lambda_{j,k}$ -minimality does imply other $\lambda_{p,q}$ -minimality. Consider the following result in [5].

Lemma 2. *The following are true for any graph G .*

- (1) $\lambda_{cj,ck}(G) = c\lambda_{j,k}$ for all positive integers c and $j \geq k$.
- (2) $\lambda_{j,k}(G) \leq \lambda_{p,q}(G)$ for all positive integers $j \geq k$ and $p \geq q$ satisfying $j \leq p$ and $k \leq q$.

We then have

Theorem 3. *Suppose G is a graph of maximum degree $\Delta \geq 1$; and $j \geq k$ and $p \geq q$ are positive integers satisfying $p = \lfloor qj/k \rfloor$. If G is $\lambda_{j,k}$ -minimal, then G is $\lambda_{p,q}$ -minimal.*

Proof. According to Lemma 2, we have

$$k\lambda_{p,q}(G) = \lambda_{kp,kq} \leq \lambda_{qj,qk} = q\lambda_{j,k} = q(j + (\Delta - 1)k).$$

Hence, $\lambda_{p,q}(G) \leq qj/k + (\Delta - 1)q$ implying $\lambda_{p,q}(G) \leq p + (\Delta - 1)q$ and so $\lambda_{p,q}(G) = p + (\Delta - 1)q$. This completes the proof of the theorem. \square

As seen above, not only is determining $\lambda_{j,k}(G)$ hard, but also deciding whether G is $\lambda_{j,k}$ -minimal may be not easy as it depends on both the structure of G and the values of j and k . In the rest of the paper, we restrict our attention to graphs that are $\lambda_{p,q}$ -minimal for all $p \geq q$.

We then have the following theorem.

Theorem 4. *Suppose G is a graph of maximum degree $\Delta \geq 1$ and $j \geq k$ are positive integers satisfying $j \geq \Delta k$. Then the following statements are equivalent.*

- (1) G is $\lambda_{j,k}$ -minimal.
- (2) G has a $\lambda_{j,k}$ -labeling g such that any vertex v in G , $g(v)$ is of the form $a_v j + b_v k$ with $a_v \in \{0, 1\}$ and $b_v \in \{0, 1, \dots, \Delta - 1\}$. Moreover, the following statements hold.
 - (D1) If $d_G(u, v) = 1$, then $a_u \neq a_v$. If $a_u = 0$ and $a_v = 1$, then $b_u \leq b_v$.
 - (D2) If $d_G(u, v) = 2$, then $a_u = a_v$ and $b_u \neq b_v$.
- (3) G is $\lambda_{p,q}$ -minimal for all positive integers $p \geq q$.

Proof. (1) \Rightarrow (2). Choose a $\lambda_{j,k}$ -labeling f of G .

For every vertex v in G , we may write $f(v) = a_v j + b_v k + r_v$, where a_v, b_v, r_v are non-negative integers satisfying $0 \leq b_v k + r_v < j$ and $0 \leq r_v < k$. Since $\lambda_{j,k}(G) = j + (\Delta - 1)k$ and $j \geq \Delta k$, we have $a_v = 0$ or 1 .

For the case where $a_v = 1$, it is clear that $0 \leq b_v \leq \Delta - 1$. For the case where $a_v = 0$, choose a vertex u adjacent to v . As $f(v) = b_v k + r_v < j$, it must be the case that $f(v) < f(u)$ and so $f(v) \leq f(u) - j \leq (\Delta - 1)k$ implying $b_v \leq \Delta - 1$. If such u does not exist, then we may assume that $f(v) = 0$. In any case, we have $0 \leq b_v \leq \Delta - 1$.

Now, consider the function g on $V(G)$ defined by $g(v) = a_v j + b_v k$ for all vertices v in G . We shall check that g is an $L(j, k)$ -labeling of G as desired.

Suppose u and v are adjacent in G . As $|f(u) - f(v)| \geq j \geq \Delta k$, it must be the case that $a_u \neq a_v$ for otherwise $|f(u) - f(v)| = |(a_u k + r_u) - (a_v k + r_v)| < \Delta k$ is impossible. Without loss of generality, we may assume that $a_u = 0$ and $a_v = 1$. In this case, $f(u) < f(v)$ and so in fact $f(u) \leq f(v) - j$ implying $b_u \leq b_v$. Then $g(u) \leq g(v) + j$.

Next, suppose $d_G(u, v) = 2$. Choose a vertex w adjacent to both u and v . Then, $a_u \neq a_w$ and $a_w \neq a_v$ implying that $a_u = a_v$. So, we have $k \leq |f(u) - f(v)| = |b_u k + r_u - b_v k - r_v|$ which implies $b_u \neq b_v$. Then $|g(u) - g(v)| \geq k$.

(2) \Rightarrow (3). Define a labeling $h : V(G) \rightarrow \{0, 1, 2, \dots\}$ by $h(v) = a_v p + b_v q$ for all vertices v in G . If $d_G(u, v) = 1$, then (D1) implies $|h(u) - h(v)| \geq p$; if $d_G(u, v) = 2$, then (D2) implies $|h(u) - h(v)| \geq q$. Hence, h is an $L(p, q)$ -labeling of G with $\max_{v \in V(G)} h(v) \leq p + (\Delta - 1)q$. Thus, G is $\lambda_{p,q}$ -minimal.

(3) \Rightarrow (1). This follows by taking $p = j$ and $q = k$. \square

Corollary 5. *If G is a $\lambda_{j,k}$ -minimal graph of maximum degree $\Delta \geq 1$ and $j \geq \Delta k$, then G is bipartite.*

Proof. Notice that in Theorem 4, every vertex has a label of the form $a_v j + b_v k$ uniquely. According to condition (D1), $a_u \neq a_v$ whenever uv is an edge. Thus, G is a bipartite graph with a bipartition of $V(G)$ into $A = \{v : a_v = 0\}$ and $B = \{v : a_v = 1\}$. \square

It is desirable to design an efficient algorithm to decide whether a graph is $\lambda_{p,q}$ -minimal for all positive integers $p \geq q$. We are only able to do this for trees. This will be discussed in the next section.

3. $\lambda_{j,k}$ -minimal trees

We now turn our attention to the case where the graph is a tree. For this case, an alternative statement of Theorem 4 is as follows.

For any positive integer Δ , a Δ -sequence is a sequence (b_0, b_1, \dots, b_m) of integers satisfying (S1) to (S4).

(S1) $b_0 = 0$.

(S2) $0 \leq b_i \leq \Delta - 1$ for all i .

(S3) $b_i \geq b_{i-1}$ and $b_i \geq b_{i+1}$ for all odd i .

(S4) $b_i \neq b_{i+2}$ for all i .

We now define T_Δ as the infinite tree (except that T_1 is K_2) whose vertex set consisting all Δ -sequences and a vertex (b_0, b_1, \dots, b_m) is adjacent to another vertex (c_0, c_1, \dots, c_n) if and only if $|m - n| = 1$ and $b_i = c_i$ for $0 \leq i \leq \min\{m, n\}$. Now, we have

Theorem 6. *Suppose T is a tree of maximum degree $\Delta \geq 1$ and $j \geq k$ are positive integers satisfying $j \geq \Delta k$. Then, T is $\lambda_{j,k}$ -minimal if and only if T is a subtree of T_Δ .*

Proof. (\Rightarrow) By Theorem 4, T has a $\lambda_{j,k}$ -labeling g such that any vertex v is labeled by $g(v) = a_v j + b_v k$ where $a_v \in \{0, 1\}$ and $b_v \in \{0, 1, \dots, \Delta - 1\}$ satisfying conditions (D1) and (D2) in Theorem 4. We may view T as a tree rooted at a major vertex v_0 . Without loss of generality, we may assume $g(v_0) = 0$ by the Proposition 1. For any vertex v in T , there is a unique path from v_0 to v , denoted by $P : v_0, v_1, \dots, v_m$. According to (D1) and (D2), $(b_{v_0}, b_{v_1}, \dots, b_{v_m})$ is a Δ -sequence. Then there is a one-to-one function from $V(T)$ to $V(T_\Delta)$ mapping v to the corresponding Δ -sequence $(b_{v_0}, b_{v_1}, \dots, b_{v_m})$. It is clear that this mapping is edge-preserving. Hence T is a subtree of T_Δ .

(\Leftarrow) According to Proposition 1, $\lambda_{j,k}(T) \geq j + (\Delta - 1)k$. Since T is a subtree of T_Δ , it suffices to give an $L(j, k)$ -labeling of T_Δ whose span is $j + (\Delta - 1)k$. Define f as follows: for any vertex (b_0, b_1, \dots, b_m) in T_Δ , let

$$f(b_0, b_1, \dots, b_m) = \begin{cases} j + b_m k, & \text{if } m \text{ is odd;} \\ b_m k, & \text{if } m \text{ is even.} \end{cases}$$

By conditions (S1) to (S4) for the definition of a Δ -sequence, it is straightforward to check that f is an $L(j, k)$ -labeling of T_Δ . \square

Note that we may determine whether a tree T of maximum degree $\Delta \geq 1$ is $\lambda_{p,q}$ -minimal for all positive integers $p \geq q$ or not by just checking whether it is $\lambda_{\Delta,1}$ -minimal. This can be done by finding $\lambda_{\Delta,1}(T)$ using the algorithm described in [1]. We are expecting a simpler algorithm from just checking whether T is a subtree of T_Δ .

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