Analytic conti of QH under ordinary flops

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<u>ordinary flops</u>. ($\mathbb{P}^r$ flop)

$$\begin{tikzcd}
\mathbb{P}^r \hookrightarrow P_S(F) = Z \subset X \arrow[r, dashed, "f"] \arrow[d, "\psi"] \arrow[dr, phantom, ""] & X' \arrow[dl, "\psi'"] \\
S \hookrightarrow \bar{X} &
\end{tikzcd}$$

with $Z \xrightarrow{\bar\psi} S$.

$\dim X = \dim S + (2r+1)$

let $\ell = [C] \in NE(X)$ be extr. ray, $C \cong \mathbb{P}^1$

with $N_{Z/X}\big|_{\bar\psi^{-1}(s)} \cong \mathcal{O}(-1)^{r+1}$

$$\begin{tikzcd}
Y = Bl_Z X \arrow[r] \arrow[d] & X' \supset Z' = P_S(F') \\
X &
\end{tikzcd}$$

for some $F' \to S$

Then $N_{Z/X} \cong \bar\psi^* F' \otimes \mathcal{O}_Z(-1)$

f simple flop if $S = pt$, Atiyah flop = simple $\mathbb{P}^1$ flop (for 3-folds)

<u>Fact</u>: ① $\mathcal{F} = [\bar\Gamma_f]_* : H(X) \xrightarrow{\sim} H(X')$, $\bar\Gamma_f = Y$

and $(\mathcal{F}a, \mathcal{F}b)^{X'} = (a,b)^X$ Poincaré pairing

② $\mathcal{F}\ell = -\ell'$. but $\mathcal{F}a . \mathcal{F}b \neq \mathcal{F}(a.b)$ in general

<u>Thm (*, Lee, Lin, W)</u> $\mathcal{F} : QH(X) \xrightarrow{\sim} QH(X')$

preserving big quantum product up to analytic conti.

for split $\mathbb{P}^r$ flops. (i.e. $F = \oplus L_i$, $F' = \oplus L_i'$).

<u>History</u>:

Aspinwall-Morrison, Witten ~'92: <u>local</u> Atiyah flop

A. Li - Y. Ruan '98: <u>global</u> Atiyah flop (degeneration formula)

LLW '06: Simple flops (in all dim)

$(QH(X), *) \leftrightarrow$ GW on $g=0$, $n \geq 3$

$$F^X(t) := \sum \frac{q^\beta}{n!} \langle t^n \rangle^X_{n,\beta} = \sum_{\substack{n\geq 0 \\ \beta \in NE(X)}} \frac{q^\beta}{n!} \int_{[\bar M_n(X,\beta)]^{vir}} \prod_{i=1}^n e_i^* t$$

$e_i : \bar M_n(X,\beta) \to X$.

a formal function on $\omega \in K^{\mathbb C}_X$ via $q^\beta = e^{2\pi i (\beta,\omega)}$

formally, $\mathcal{F} q^\beta = q^{\mathcal F \beta}$

$F^X(t)$, $F^{X'}(t)$ defined on same $t \in H = H(X) \overset{\mathcal F}{\cong} H(X')$ but diff variables in $NE(X)$, $NE(X')$

in fact $K^{\mathbb C}_X \cap K^{\mathbb C}_{X'} = \phi$, the comparision makes sense ~~or just $\mathcal F q^\ell = q^{-\ell'}$~~ only in analytic conti.

let $t = \sum t^i T_i$, $\{T_i\}$ basis of H, $\{T^i\}$ dual

$$T_i *_t T_j = \sum \frac{\partial^3 F^X}{\partial t^i \partial t^j \partial t^k}(t)\, T^k = \sum \frac{q^\beta}{n!} \langle T_i, T_j, T_k, t^n \rangle^X_{n+3,\beta} T^k$$

WDVV $\iff$ $*_t$ is a family of associative product on H, parametrized by $t \in H$.

Rmk: $\iff$ flatness of Dubrovin conn

$$\nabla^z = d - \frac{1}{z} \sum dt^i \otimes T_i *_t \quad \text{on } TH = H \times H \quad (z \in \mathbb C^\times)$$

i.e. A formal Frobenius mfd $H_X \hookleftarrow K^{\mathbb C}_X$

$\{q^\ell, q^{\ell'}\}$ serves as atlas for $P^1 = \overline{\mathbb C/\mathbb Z} \cong \overline{\mathbb C^\times}$

convergence of $QH(X)$ in q^ℓ, $QH(X')$ in $q^{\ell'}$ $\Big\}$ $\mathcal F \rightsquigarrow$ analytic $P^1 \subset H$.

in fact $\partial^3_{ijk} F^X(t)$ is algebraic in q^ℓ for simple flops.

Steps toward the pf:

(1) Defect of cup product

(2) Quantum correction attached to $\mathbb N \ell$. (GMCF)

(3) Reduction to local model by degeneration analysis.

$X_{loc} = P(N_{Z/X} \oplus \mathcal O)$, $X'_{loc} = P(N_{Z'/X'} \oplus \mathcal O)$ defined by (S, F, F')

(4) reduction to quasi-linearity via reconstruction + WDVV

(5) pf of Q-L. (for split case). via GMT + BF.

(1) $\{t_i\} \subset A(S)$ basis, $\hat{t}_i$ dual. $h = c_1(\mathcal{O}_Z(1))$

$H_k = c_k(Q_F)$ univ. quot. $\to Z = P(F)$

$$\Rightarrow \begin{cases} j^* H_k = (-1)^{r-k} H_k' \\ \{t_i h^j\} \subset A(Z) \text{ has dual } \{\hat{t}_i H_{r-j}\} \end{cases}$$

Prop 1. $a_i \in A(X)$, $\sum \deg a_i = \dim X$ (Chow degree)

$$(j^*a_1, j^*a_2, j^*a_3)^{X'} - (a_1, a_2, a_3)^X$$

$$= (-1)^r \sum_{i_*, j_*} (a_1, \hat{t}_{i_1} H_{r-j_1})^X (a_2, \hat{t}_{i_2} H_{r-j_2})^X (a_3, \hat{t}_{i_3} H_{r-j_3})^X$$

$$\times \big(\underline{s_{j_1+j_2+j_3-(2r+1)}(F+F'^*)} . t_{i_1} t_{i_2} t_{i_3} \big)^S$$

Segre class.

(2) GW moduli on $d\ell$ has bundle str$/S$:

$$\begin{tikzcd} \overline{M}_n(P^r, d\ell) \arrow[r] & \overline{M}_n(Z, d\ell) \arrow[r, "e_i"] \arrow[d, "\Psi_n"] & Z \arrow[dl, "\bar{\Psi}"] \\ & S & \end{tikzcd}$$

GW on X $\leftrightarrow$ twisted inv on Z by obst. bundle.

$$\Big\langle \prod_{i=1}^n h^{j_i} \Big\rangle_d^{/S} := \Psi_{n*}\Big(\prod_{i=1}^n e_i^* h^{j_i}\Big) \in A^{\mu}(S)$$

$\bar{\Psi}$ rel-inv$/S$. $\mu := \sum j_i - (2r+1+n-3)$

$$\Rightarrow \langle t_1 h^{j_1}, \dots, t_n h^{j_n} \rangle_d^X = \big(\langle h^{j_1}, \dots, h^{j_n} \rangle_d^{/S} . t_1 \cdots t_n \big)^S.$$

$\mu = 0 \Rightarrow$ reduce to simple case (done)

true if $n = 2$: $\Rightarrow j_1 = j_2 = r$ (since may let $\leq r$)

$$(\alpha_1, \alpha_2)^X_{2, d\ell} = \sum (\alpha_1 . t_s)(\alpha_2 . \underset{\underset{t_{s_2}}{\parallel}}{\hat{t}_s}) \langle h^r, h^r \rangle_d^{\text{simple}} \underset{\underset{1}{\parallel}}{(t_{s_1}, t_{s_2})^S}$$

$$= (-1)^{(d-1)(r+1)} \frac{1}{d} \sum_s (\alpha_1 . t_s)(\alpha_2 . \hat{t}_s).$$

- notice $\langle \alpha_1, \alpha_2 \rangle_2^X = (\alpha_1, \alpha_2)^X_{2, d\ell} q^{d\ell}$ is not j-cov. $\to \log(1 \pm q)$.

Lee-Pandharipande ('04 ?): Divisor relation / reconstruction

$$e_i^* L = e_j^* L + \sum_{\beta_1 + \beta_2 = \beta} (\beta_2 . L) [D_{ik, \beta_1 | j, \beta_2}]^{vir} - (\beta_1 . L) [D_{i \beta_1 | jk, \beta_2}]^{vir}.$$

$f(q) := \dfrac{q}{1-(-1)^{r+1}q} = q + (-1)^{r+1}q^2 + \cdots$

Basic FE: $f(q) + f(q^{-1}) = (-1)^r$. Let $\delta = q\frac{d}{dq}$ degree operator

$W_\mu := \sum_{d=1}^{\infty} \langle h^{j_1}, h^{j_2}, h^{j_3} \rangle_{\ell/S}\, q^d \qquad 1 \le j_i \le r$

$\in AM(S)$ is independent of choices of j_i's ($\mu = j_1+j_2+j_3-(2r+1)$ fixed)

Prop 2 $W_\mu \overset{\exists}{=} P_\mu(\delta) f$ chern class valued polynomial

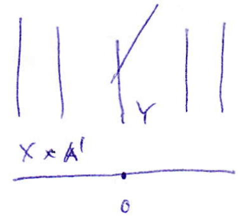
and $W_\mu - (-1)^{\mu+1} W'_\mu = (-1)^r s_\mu(F + F'^*)$.

($n=3$ done, $n \geq 4$ follows by reconstruction.)

The starting case $\langle h, h^r, h^r \rangle = \delta \langle h^r, h^r \rangle$ is clear.

(3) For non-extremal curves. $N := N_{Z/X}$ $\tilde{E} = P(N \oplus O)$

Deformation to the normal cone:

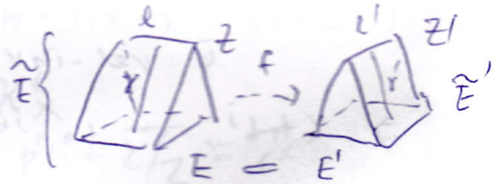


$\bar{p} : \tilde{E} = P(N \oplus O) \xrightarrow{p} Z \xrightarrow{\psi} S$

$\bar{p}' : \tilde{E}' = P(N' \oplus O) \xrightarrow{p'} Z' \xrightarrow{\psi'} S$

induced flop $f : \underset{X}{\tilde{E}} \dashrightarrow \underset{X'}{\tilde{E}'}$ a family of simple case $/S$.

$$\begin{tikzcd} N_1(\tilde{E}) \arrow[rr, "\mathcal{F}"] \arrow[dr, "\bar{p}_* \oplus d_2"'] & & N_1(\tilde{E}') \arrow[dl, "\bar{p}'_* \oplus d'_2"] \\ & N_1(S) \oplus \mathbb{Z} & \end{tikzcd}$$



(d_2 may be < 0!)

$\beta \in NE(\tilde{E})$, $\beta = $ "β_S" $+ d\ell + d_2\gamma$ "β_S" = canonical lift

double proj bundle $\psi^*\beta_S . H_r$

- $\mathcal{F}\langle A \rangle^X \simeq \langle \mathcal{F}A \rangle^{X'}$ FE / Analy conti for $\beta_S \in NE(S)$.

$\Longleftrightarrow$ $\mathcal{F}\langle A \rangle^X_{\beta_S, d_2} \simeq \langle \mathcal{F}A \rangle^{X'}_{\beta_S, d_2}$ st. "β_S"$. h = 0$.

Prop 3. To get $\mathcal{F}\langle \alpha \rangle^X \simeq \langle \mathcal{F}\alpha \rangle^{X'}$ $\forall \alpha$. ($n \geq 3$)

enough to show the local case $f : \tilde{E} \dashrightarrow \tilde{E}'$

for descendent of f-special type:

$\mathcal{F}\langle A, \tau_{k_1}\varepsilon_1, \cdots, \tau_{k_p}\varepsilon_p \rangle^{\tilde{E}}_{\beta_S, d_2} \simeq \langle \mathcal{F}A, \tau_{k_1}\varepsilon_1, \cdots, \tau_{k_p}\varepsilon_p \rangle^{\tilde{E}'}_{\beta_S, d_2}$.

and $d_2 \geq 0$.

(4) Let $X = \tilde{E}$, $X' = \tilde{E}'$,
$\xi = \cancel{c_1(\mathcal{O}_{\hat{E}})}$ hyp class of $\tilde{E} \to Z$, ξ'

$\rightsquigarrow \langle t_1, t_2, \cdots, t_n \underline{T_k} h^j \xi^i \rangle^X_{\beta_S, d_2}$.
$k \neq 0$ if $i \neq 0$.

To assume $i \neq 0$, if $d_2 \neq 0$ ok by div axiom
$d_2 = 0$ by WDVV eq'n:

$[a \vee b \mapsto \xi c \vee \xi d] = [a \vee \xi c \mapsto b \vee \xi d]$

$(\beta_S, d_2 = 1)$: $\sum \langle a, b, t_i h^j \rangle_{\beta_S, i} \underline{\langle t_i^* H_{r-j} \cdot \Theta_{r+1}, \xi c, \xi d \rangle_{0, d_2}}$

Prop 4. let $c = t_k h^j$, $d = h^r$ $= \underline{I_{c,d}}$

$$\langle h^{r-j} (\xi - h)^{r+1}, \xi^{r+1}, \xi h^r \rangle_{d_2 = 1} = \begin{cases} (-1)^j q^\ell q^\gamma & \underline{0 \le j \le r-1} \\ ((1 - (-1)^{r+1} q^\ell) q^\gamma & j = r \end{cases}$$