Ordinary Flops

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CONTENTS

1.1 Ordinary (r, r') Flips

X smooth projective,
$\psi : X \to \bar{X}$ log-extremal small contraction,
$R = \mathbb{R}^{+}[C]$, the log-extremal ray,
$Z \subset X$ and $S \subset \bar{X}$: ψ exceptional sets,
$\bar{\psi} = \psi|_Z : Z \to S$, $Z_s := \bar{\psi}^{-1}(s)$.

ψ is a (r, r') flipping contraction if
(i) $\bar{\psi} : Z = \mathbb{P}_S(F) \to S$ for some rank $r + 1$ vector bundle F over a smooth base S,
(ii) $N_{Z/X}|_{Z_s} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r'+1)}$.

Fact: Let $\bar{\psi} : Z = \mathbb{P}_S(F) \to S$ and $V \to Z$ a vector bundle such that $V|_{Z_s}$ is trivial $\forall s \in S$. Then $V \cong \bar{\psi}^* F'$ for some vector bundle F'.

Apply to $V = \mathcal{O}_{\mathbb{P}_S(F)}(1) \otimes N_{Z/X}$, we get

$$N_{Z/X} \cong \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^* F'.$$

Since $\mathcal{O}_{\mathbb{P}_Z(L \otimes F)}(-1) = \bar{\phi}^* L \otimes \mathcal{O}_{\mathbb{P}_Z(F)}(-1)$ for $L \in \mathrm{Pic}(Z)$, on the blow-up $\phi : Y = \mathrm{Bl}_Z X \to X$,

$$E = \mathbb{P}_Z(N_{Z/X}) \cong \mathbb{P}_Z(\bar{\psi}^* F') = \bar{\psi}^* \mathbb{P}_S(F') = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F'),$$

$$N_{E/Y} = \mathcal{O}_{\mathbb{P}_Z(N_{Z/X})}(-1) = \bar{\phi}^* \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\phi}'^* \mathcal{O}_{\mathbb{P}_S(F')}(-1).$$

Basic diagram: $g = \psi \circ \phi : Y \to \bar{X}$, $\bar{g} = g|_E$,

$$\begin{array}{ccccc} & & E = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F') \subset Y & & \\ & \swarrow^{\bar{\phi}} & \downarrow \bar{g} \, \subset g & \searrow^{\bar{\phi}'} & \\ Z = \mathbb{P}_S(F) \subset X & & & & Z' = \mathbb{P}_S(F') \\ & \searrow_{\bar{\psi}} & & \swarrow_{\bar{\psi}'} & \\ & & S \subset \bar{X} & & \end{array}$$

The pair (F, F') is unique up to a twisting: $(F, F') \sim (F \otimes L, F' \otimes L^*)$ for all $L \in \mathrm{Pic}(S)$.

Theorem 1 *Ordinary (r, r')-flip $f : X \dashrightarrow X'$ exists. Moreover, $Y = \bar{\Gamma}_f = X \times_{\bar{X}} X' \subset X \times X'$.*

Proof. From $0 \to T_C \to T_X|_C \to N_{C/X} \to 0$ and $N_{C/X} \cong \mathcal{O}_C(1)^{\oplus(r-1)} \oplus \mathcal{O}_C(-1)^{\oplus(r'+1)}$,

$$K_X.C = 2g(C) - 2 - ((r-1) - (r'+1)) = r' - r.$$

Pick a line $C_Y \in \bar{\phi}^{-1}(\text{pt})$, $\phi(C_Y) = C$. Then

$$K_Y.C_Y = (\phi^* K_X + r'E).C_Y = (r' - r) - r' = -r < 0.$$

Let H be very ample on X and L a supporting divisor of C. Let $c = H.C$. For large k,

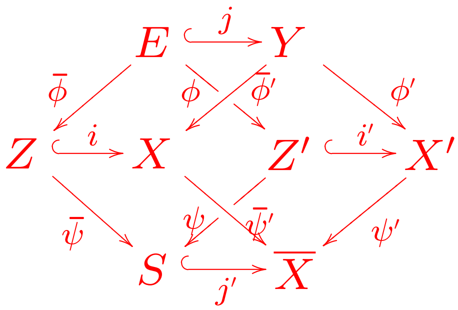
$$k\phi^* L - (\phi^* H + cE)$$

is big and nef, and vanishes precisely on $[C_Y]$. Thus C_Y is a K_Y-negative extremal ray and $\phi' : Y \to X'$ exists by the cone theorem. $\square$

1.2 Analytic Local Models

$F \to S$, $F' \to S$: holomorphic vector bundles,
$\bar\psi : Z = \mathbb{P}_S(F) \to S$, $\bar\psi' : Z' = \mathbb{P}_S(F') \to S$,
$E = Z \times_S Z'$ with projections $\bar\phi$ and $\bar\phi'$.
$Y =$ total space of $N := \bar\phi^* \mathcal{O}_Z(-1) \otimes \bar\phi'^* \mathcal{O}_{Z'}(-1)$,
$E =$ zero section, $N_{E/Y} = N$.

We have analytic contraction diagram



X and X' are smooth, $S = \mathrm{Sing}(\bar{X})$.

$$X = \text{total space of } N_{Z/X} = \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^* F',$$

$$X' = \text{total space of } N_{Z'/X'} = \mathcal{O}_{\mathbb{P}_S(F')}(-1) \otimes \bar{\psi}'^* F.$$

Again, (F, F') and (F_1, F'_1) define isomorphic analytic local model if and only if $(F_1, F'_1) = (F \otimes L, F' \otimes L^*)$ for some $L \in \mathrm{Pic}(S)$.

An ordinary (r, r)-flip is called an ordinary $\mathbb{P}^r$ flop or simply a $\mathbb{P}^r$ flop.

2.1 Canonical Correspondences

$\mathcal{M}$: category of motives. Objects = smooth varieties, morphisms = correspondences

$$\mathrm{Hom}_{\mathcal{M}}(\hat{X}_1, \hat{X}_2) = A^*(X_1 \times X_2)$$

under composition law: for $U \in A^*(X_1 \times X_2)$, $V \in A^*(X_2 \times X_3)$, $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$,

$$V \circ U = p_{13*}(p_{12}^* U . p_{23}^* V),$$

$$[U] : A^*(X_1) \to A^*(X_2); \quad a \mapsto p_{2*}(U . p_1^* a).$$

Induced map on T-valued points $\mathrm{Hom}(\hat{T}, \hat{X}_i)$:

$$U_T : A^*(T \times X_1) \xrightarrow{U\circ} A^*(T \times X_2).$$

Identity Principle: Let $U, V \in \mathrm{Hom}(\hat{X}, \hat{X}')$. Then $U = V$ if and only if $U_T = V_T$ for all T. $(U_X \circ \Delta_X = V_X \circ \Delta_{X'}$ implies $U = V$.)

Theorem 2 *For ordinary flops $f : X \dashrightarrow X'$, the graph closure $\mathfrak{F} := \bar{\Gamma}_f$ induces $\hat{X} \cong \hat{X}'$ via $\mathfrak{F}^* \circ \mathfrak{F} = \Delta_X$ and $\mathfrak{F} \circ \mathfrak{F}^* = \Delta_{X'}$.*

Proof. For any T, $\mathrm{id}_T \times f : T \times X \dashrightarrow T \times X'$ is also an ordinary flop. By the identity principle we only need to show $\mathcal{F}^*\mathcal{F} = \mathrm{id}$ on $A^*(X)$.

$$\mathcal{F}W = p'_*(\bar{\Gamma}_f.p^*W) = \phi'_*\phi^*W.$$

$$\phi^*W = \tilde{W} + j_*\Big(c(\mathcal{E}).\bar{\phi}^*s(W \cap Z, W)\Big)_{\dim W},$$

where $0 \to N_{E/Y} \to \phi^*N_{Z/X} \to \mathcal{E} \to 0$ and $s(W \cap Z, W)$ is the relative Segre class.

Observation: the error term is lying over $W \cap Z$.

$$\begin{array}{ccccc} \mathbb{P}^r \times \mathbb{P}^r & \longrightarrow & E^{2r+s} & \hookrightarrow & Y^{2r+s+1} \\ & & \downarrow & & \\ \mathbb{P}^r & \longrightarrow & Z^{r+s} & & \\ & & \downarrow & & \\ & & S^s & & \end{array}$$

Let $W \in A_k(X)$. By Chow's moving lemma we may assume that W intersects Z transversally. $\dim W \cap Z := \ell = k + (r+s) - (2r+s+1) = k - r - 1$. Since $\dim \phi^{-1}(W \cap Z) = \ell + r < k$, we get $\phi^* W = \tilde{W}$ and $\phi^{-1}(W) \cap E = \phi^{-1}(W \cap Z)$. Hence $\mathfrak{F}W = W'$, the proper transform of W in X'. (W' may not be transversal to Z'.)

Let B be an irreducible component of $W \cap Z$ and $\bar{B} = \bar{\psi}(B) \subset S$ with dimension $\ell_B \leq \ell$. Notice that $W' \cap Z'$ has irreducible components $\{\bar{\psi}'^{-1}(\bar{B})\}_B$. Let $\phi'^* W' = \tilde{W} + \sum_B E_B$.

$E_B \subset \bar{\phi}'^{-1}\bar{\psi}^{-1}(\bar{B})$, a $\mathbb{P}^r \times \mathbb{P}^r$ bundle over $\bar{B}$. For the generic point $s \in \bar{B}$, we thus have $\dim E_{B,s} \geq k - \ell_B = r + 1 + (\ell - \ell_B) > r = \frac{1}{2} 2r$.

In particular, $E_{B,s}$ contains positive dimensional fibers of ϕ and ϕ' and $\phi_*(E_B) = 0$. So $\mathcal{F}^*\mathcal{F}W = W$. The proof is completed. $\square$

2.2 The Poincaré Pairing

Corollary 3 *Let $f : X \dashrightarrow X'$ be an ordinary flop. If* $\dim \alpha + \dim \beta = \dim X$*, then*

$$(\mathfrak{F}\alpha.\mathfrak{F}\beta) = (\alpha.\beta).$$

That is, $\mathfrak{F}$ is orthogonal with respect to $(-.-)$.

Proof. $\alpha.\beta = \phi^*\alpha.\phi^*\beta = (\phi'^*\mathfrak{F}\alpha + \xi).\phi^*\beta = (\phi'^*\mathfrak{F}\alpha).\phi^*\beta = \mathfrak{F}\alpha.(\phi'_*\phi^*\beta) = \mathfrak{F}\alpha.\mathfrak{F}\beta.$ □

Remark: $\mathfrak{F}^{-1} = \mathfrak{F}^*$ both in the sense of correspondences and Poincaré pairing.

3.1 Triple Product for Simple Flops

$f : X \dashrightarrow X'$ a simple $\mathbb{P}^r$ flop, $S = \text{pt}$,
$h =$ hyperplane class of $Z = \mathbb{P}^r$,
$h' =$ hyperplane class of Z',
$x = [h \times \mathbb{P}^r]$, $y = [\mathbb{P}^r \times h']$ in $E = \mathbb{P}^r \times \mathbb{P}^r$.

$$\phi^*[h^s] = x^s y^r - x^{s+1} y^{r-1} + \cdots + (-1)^{r-s} x^r y^s,$$

$$\mathcal{F}[h^s] = (-1)^{r-s}[h'^s],$$

$$\phi'^*\alpha' = \phi^*\alpha + (\alpha.h^{r-i})\,\frac{x^i + (-1)^{i-1} y^i}{x+y}, \qquad \alpha \in A^i(X).$$

Theorem 4 *For simple $\mathbb{P}^r$-flops, $\alpha \in A^i(X)$, $\beta \in A^j(X)$, $\gamma \in A^k(X)$ with $i \le j \le k \le r$, $i + j + k = \dim X = 2r + 1$,*

$$\mathfrak{F}\alpha.\mathfrak{F}\beta.\mathfrak{F}\gamma = \alpha.\beta.\gamma + (-1)^r(\alpha.h^{r-i})(\beta.h^{r-j})(\gamma.h^{r-k}).$$

Example: $r = 2$, $\dim X = 5$, $(i, j, k) = (1, 2, 2)$:

$$\begin{aligned} T\alpha.T\beta.T\gamma &= \alpha'.\beta'.\gamma' = \phi'^*\alpha'.\phi'^*\gamma'.\phi^*\gamma' \\ &= (\phi^*\alpha + (\alpha.h)E)(\phi^*\beta + (\beta.Z)(x-y))(\phi^*\gamma + (\gamma.Z)(x-y)) \\ &= \alpha.\beta.\gamma + (\beta.Z)(\gamma.Z)\phi^*\alpha.(x-y)^2 \\ &\qquad + (\alpha.h)(\gamma.Z)\phi^*\beta.E.(x-y) + (\alpha.h)(\beta.Z)\phi^*\gamma.E.(x-y) \\ &\qquad + (\alpha.h)(\beta.Z)(\gamma.Z)E.(x-y)^2 \\ &= \alpha.\beta.\gamma + (\alpha.h)(\beta.Z)(\gamma.Z). \end{aligned}$$

3.2 Quantum Corrections (Outline)

The three point functions

$$\begin{aligned}\langle\alpha,\beta,\gamma\rangle &= \sum_{d\in A_1(X)} \langle\alpha,\beta,\gamma\rangle_{0,3,d} \\ &= \alpha.\beta.\gamma + \sum_{k\in\mathbb{N}} \langle\alpha,\beta,\gamma\rangle_{0,3,k[C]}\, q^{k[C]} \\ &\quad + \sum_{d\neq k[C]} \langle\alpha,\beta,\gamma\rangle_{0,3,d}\, q^{d}\end{aligned}$$

and $(-,-)$ determine the quantum product. The difference of $\alpha.\beta.\gamma$ is already determined.

Deformations to the normal cone: $\mathcal{X} = X \times \mathbb{P}^1$, $\Phi : M \to \mathcal{X}$ be the blowing-up along $Z \times \{\infty\}$. $M_t \cong X$ for all $t \neq \infty$ and $M_\infty = Y \cup \tilde{E}$ where $\tilde{E} = \mathbb{P}_S(N_{Z/X} \oplus \mathcal{O})$. $Y \cap \tilde{E} = E = \mathbb{P}_S(N_{Z/X})$ is the infinity part of $\tilde{E}$. Similarly $\Phi' : M' \to \mathcal{X}' = X' \times \mathbb{P}^1$ and $M'_\infty = Y' \cup \tilde{E}'$. $Y = Y'$ and $E = E'$.

When $S = $ pt, $\tilde{E} \cong \tilde{E}'$. J. Li's degeneration formula (A. Li and Y. Ruan) implies the equivalence of $\langle \alpha, \beta, \gamma \rangle_{0,3,d}$ with $d \neq k[C]$.

For simple $\mathbb{P}^1$-flops, the second term gives

$$\sum_k (\alpha.k[C])(\beta.k[C])(\gamma.k[C]) \left\langle I_{0,0,k[C]} \right\rangle q^{k[C]}.$$

$$= (\alpha.C)(\beta.C)(\gamma.C)\frac{q^{[C]}}{1 - q^{[C]}}$$

by the multiple cover formula (Voisin).

For simple $\mathbb{P}^2$-flops of type $(1, 2, 2)$,

$$\langle \alpha, \beta, \gamma \rangle_{0,3,k[C]} = k(\alpha.C)(\beta.Z)(\gamma.Z) \times \int_{\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2,k)} c_{3(k-1)}(R^1\pi_* e_3^* \mathcal{O}(-1)^{\oplus 3}),$$

with $e_3 : \overline{\mathcal{M}}_{0,3}(\mathbb{P}^2, k) \to X$ and $\pi : \overline{\mathcal{M}}_{0,3}(\mathbb{P}^2, k) \to \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, k)$. (Work in progress.)

3.3 Some Explicit Formulae

For $\mathbb{P}^r$-flop with non-trivial base S, $\alpha \in A^*(Z)$ has the form $\alpha = \sum \xi^i \bar{\psi}^* a_i$; $\xi = c_1(\mathcal{O}_{\mathbb{P}(F)}(-1))$, $a_i \in A^*(S)$. $\mathcal{E} = \bar{\phi}^* \mathcal{O}_{\mathbb{P}(F)}(-1) \otimes \bar{\phi}'^* \mathcal{Q}_{F'}$.

$$\mathcal{F}\alpha = \sum \mathcal{F}(\xi^i).\bar{\psi}'^* a_i = \sum \bar{\phi}'_*(c_r(\mathcal{E}).\bar{\phi}^* \xi^i).\bar{\psi}'^* a_i.$$

$$\mathcal{F}\xi^1 = (-1)^{r-1}(\xi' - \bar{\psi}^*[c_1(F) + c_1(F')]).$$
$$\mathcal{F}\xi^2 = (-1)^{r-2}(\xi'^2 - \bar{\psi}^*[(c_1 + c'_1).\xi' + (c_1^2 + c_1 c'_1 - c_2 + c'_2)]).$$
$$\mathcal{F}\xi^3 = (-1)^{r-3}(\xi'^3 - \bar{\psi}'^*[(c_1 + c'_1)\xi'^2 + (c_1^2 + c_1 c'_1 - c_2 + c'_2)\xi' + (c_1^3 - 2c_1 c_2 - c_2 c'_1 + c_1^2 c'_1 + c_1 c'_2 + c_3)]).$$

4.1 Deformations

Theorem 5 *Ordinary flips deform in families: let* $f : X \dashrightarrow X'$ *be an* (r, r') *flip with base* S *and* $\mathcal{X} \to \Delta$ *be a smooth family with* $\mathcal{X}_0 = X$*. Then there is a smooth family* $\mathcal{X}' \to \Delta$ *and a* Δ*-birational map* $F : \mathcal{X} \dashrightarrow \mathcal{X}'$ *such that* $F_0 = f$*. Moreover,* F *is also an* (r, r') *flip, with base* $\mathcal{S} \to \Delta$ *an one parameter deformations of* S*.*

Key: the ray $[C]$ is stable in deformations.

Idea. $\text{Hilb}_{C/X}$ is a $G(2, r+1)$ bundle over S.

$$N_{C/\mathcal{X}} \cong \mathcal{O}(1)^{\oplus(r-1)} \oplus \mathcal{O}(-1)^{\oplus(r'+1)} \oplus \mathcal{O}^{s+1}.$$

$H^1(C, \mathcal{O}(k)) = 0$ for all $k \geq -1$ implies that $\text{Hilb}_{C/\mathcal{X}}$ is smooth at $[C]$ for all $C \subset Z$ and the natural map $\pi : \text{Hilb}_{C/\mathcal{X}} \to \Delta$ is a smooth fibration with special fiber $\text{Hilb}_{C/X}$. By the stability of Grassmannian bundles we obtain $\mathcal{Z} \to \mathcal{S} \to \Delta$. The supporting line bundles $\mathcal{L}$ for C on $\mathcal{X}$ is the unique extension of the supporting line bundle L for C on X. □

4.2 Degenerations

Fact. Every three dimensional smooth flop is the limit of composite of $\mathbb{P}^1$ flops.

Question: What is the closure of composite of general ordinary flops?

5.1 Generalized Mukai Flops

$\psi : (X, Z) \to (\bar{X}, S)$ with $N_{Z/X} = T^*_{Z/S} \otimes \bar{\psi}^* L$, $L \in \mathrm{Pic}(S)$. Will construct the local model as a section of ordinary flops with $F' = F^* \otimes L$.

$$\begin{array}{ccccc} & & \mathcal{E} = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F') \subset \mathcal{Y} & & \\ & \swarrow^{\Phi} & \downarrow g & \searrow^{\Phi'} & \\ Z = \mathbb{P}_S(F) \subset \mathcal{X} & & & & Z' = \mathbb{P}_S(F') \subset \mathcal{X}' \\ & \searrow_{\Psi} & & \swarrow_{\Psi'} & \\ & & S \subset \bar{\mathcal{X}} & & \end{array}$$

Suppose $\exists$ bi-linear map $F \times_S F' \to \eta_S$ to a line bundle η_S over S. $\mathcal{O}_{\mathbb{P}(F)}(-1) \to \bar{\psi}^* F$ pulls back to $\bar{\phi}^* \mathcal{O}_{\mathbb{P}(F)}(-1) \to \bar{g}^* F$, hence a linear map

$$\bar{\phi}^* \mathcal{O}_Z(-1) \otimes_{\mathcal{E}} \bar{\phi}'^* \mathcal{O}_{Z'}(-1) \to \bar{g}^*(F \otimes_S F') \to \bar{g}^* \eta_S.$$

$Y :=$ inverse image of the zero section of $\bar{g}^* \eta_S$ in $\mathcal{Y}$. $X = \Phi(Y) \supset Z$, $X' = \Phi'(Y) \supset Z'$, $\bar{X} = g(Y) \supset S$ with restriction maps ϕ, ϕ', ψ, ψ'. By tensoring the Euler sequence

$$0 \to \mathcal{O}_Z(-1) \to \bar{\psi}^* F \to \mathcal{Q} \to 0$$

with $\mathcal{S}^* = \mathcal{O}_Z(1)$ and notice that $\mathcal{S}^* \otimes \mathcal{Q} \cong T_{Z/S}$, we get by dualization

$$0 \to T^*_{Z/S} \to \mathcal{O}_Z(-1) \otimes \bar{\psi}^* F^* \to \mathcal{O}_Z \to 0.$$

The inclusion maps $Z \hookrightarrow X \hookrightarrow \mathcal{X}$ leads to

$$0 \to N_{Z/X} \to N_{Z/\mathcal{X}} \to N_{X/\mathcal{X}}|_Z \to 0.$$

$N_{X/\mathcal{X}}|_Z = \mathcal{O}(X)|_Z = \bar{\psi}^*\mathcal{O}(\bar{X})|_S$. Denote $\mathcal{O}(\bar{X})|_S$ by L. Recall $N_{Z/\mathcal{X}} \cong \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^* F'$. By tensoring with $\bar{\psi}^* L^*$, we get

$$0 \to N_{Z/X} \otimes \bar{\psi}^* L^* \to \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^*(F' \otimes L^*) \to \mathcal{O}_Z \to 0.$$

So $F' = F^* \otimes L$ if and only if $N_{Z/X} \cong T^*_{Z/S} \otimes \bar{\psi}^* L$.

5.2 Mukai Flops as Limits of Isomorphisms

For Mukai flops, $L \cong \mathcal{O}_S$, $F' = F^*$ with duality pairing $F \times_S F^* \to \mathcal{O}_S$. Consider $\pi : \mathcal{Y} \to \mathbb{C}$ via

$$\mathcal{Y} \to \bar{g}^* \mathcal{O}_S = \mathcal{O}_{\mathcal{E}} \cong \mathcal{E} \times \mathbb{C} \xrightarrow{\pi_2} \mathbb{C}.$$

We get a fibration with $\mathcal{Y}_t := \pi^{-1}(t)$, being smooth for $t \neq 0$ and $\mathcal{Y}_0 = Y \cup \mathcal{E}$. $E = Y \cap \mathcal{E}$ restricts to the degree $(1, 1)$ hypersurface over each fiber along $\mathcal{E} \to S$. Let $\mathcal{X}_t$, $\mathcal{X}'_t$ and $\bar{\mathcal{X}}_t$ be the proper transforms of $\mathcal{Y}_t$ in $\mathcal{X}$, $\mathcal{X}'$ and $\bar{\mathcal{X}}$.

For $t \neq 0$, all maps in the diagram

$$
\begin{array}{ccccc}
 & & \mathcal{Y}_t & & \\
 & \swarrow & & \searrow & \\
\mathcal{X}_t & & & & \mathcal{X}'_t \\
 & \searrow & & \swarrow & \\
 & & \bar{\mathcal{X}}_t & &
\end{array}
$$

are all isomorphisms. For $t = 0$ this is the Mukai flop. Thus Mukai flops are limits of isomorphisms. They preserve all interesting invariants like diffeomorphism type, Hodge type (Chow motive via $[Y] + [\mathcal{E}]$) and quantum rings etc. In fact all quantum corrections are zero.