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NCTS (北區), Workshop on HD Alg. Geom

Quasi-Hodge metrics and canonical singularities

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Filling-in Problem:

$$\begin{tikzcd} \mathcal{X}^\times \ \text{proj. sm} \arrow[d] & & \mathcal{X} \supset \mathcal{X}_0 \ \text{proj} \arrow[d] \\ \Delta^\times \arrow[rr, hook] & & \Delta \end{tikzcd}$$

when can $\mathcal{X}_0$ be smooth? irreducible? irreducible with mild can. sing? e.g. (up to a finite base change on $\Delta^\times$)

GOAL: To get criterions which depend only on "diff-geometric" data on $\Delta^\times$.

for $\begin{matrix}\mathcal{X} \\ \downarrow \pi \\ S\end{matrix}$ sm. proj $p_g(\mathcal{X}_s) \neq 0$ the Hodge bundle $F^n = \pi_* K_{\mathcal{X}/S}$

has semi-pos 1st chern form (Griffiths), which defines the quasi-Hodge metric $g_H = g_1$ on S. (possibly degenerate)

When $S = \Delta^\times$, let T = monodromy on $H^n(\mathcal{X}_s, \mathbb{C})$

Thm 1. Let $\pi : \mathcal{X}^\times \to \Delta^\times$ sm. proj $n = \dim \mathcal{X}_s$,

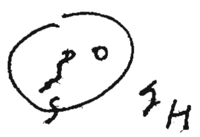
(1) g_H incomplete $\Longleftrightarrow$ $T^r = id$ on $H^{n,0}(\mathcal{X}_s)$ for some $r \in \mathbb{N}$.

(2) for a semi-stable model $\mathcal{X} \to \Delta$, $\mathcal{X}_0 = \bigcup_{i=0}^N X_i$, $\Longleftrightarrow$ $p_g(\mathcal{X}_s) = \sum_{i=0}^N p_g(X_i)$

(3) Degeneration with Gorenstein can sing $\Rightarrow$ g_H incomplete.

Thm 2. Let $\mathcal{X} \to \Delta$ degeneration of C-Y n-folds st $0 \in \Delta$ is at finite g_H distance (i.e. incomplete)

Then MMC in dim $= n+1$ $\Rightarrow$ up to a finite base change on Δ,



$$\begin{tikzcd} \mathcal{X} \arrow[r, dashed] \arrow[dr] & \mathcal{X}' \arrow[d] \\ & \Delta \end{tikzcd}$$

isom over Δ^* and $\mathcal{X}_0'$ is C-Y with at most can. sing.

the rel. min model will do.

Idea: Take $\mathcal{X} \to \Delta$ s.s. By thm 1-(2)

$1 = p_g(\mathcal{X}_s) = \sum_i p_g(X_i)$, say $p_g(X_0) = 1$, $p_g(X_i) = 0 \ \forall i \neq 0$

$1 = P_m(\mathcal{X}_s) \geq \sum_i P_m(X_i)$

$\Rightarrow P_m(X_0) = 1$, $P_m(X_i) = 0 \quad \forall m \geq 1$.

X_i should be uni-ruled and contractible.

Rmk: For C-Y, $g_H = g_{WP}$ (Weil-Petersson) defined via Ricci flat metrics on $\mathcal{X}_s$.

In general for $p_g(\mathcal{X}_s) > 1$, g_H incomplete is insufficient to characterize "Canonical degenerations". Need to "Concentrate all P_m in one component"!

The m-th quasi-Hodge metric g_m:

consider $\begin{matrix}\mathcal{X}\\ \downarrow \\ S\end{matrix}$ with $\mathcal{X}_s$ semi-ample, $\underset{\text{sm}}{\mathcal{D}} \in |K_{\mathcal{X}/S}^{\otimes m}|$ for m large.

Take $\begin{matrix}\mathcal{Y}\\ \downarrow \\ S\end{matrix}$ the m-cyclic cover of $\mathcal{X}/S$ along $\mathcal{D}$.

Then $g_m := g_H$ wrt $\mathcal{Y}/S$.

Thm 3. Let $\pi: \mathcal{X} \to \Delta$ degeneration of sm proj mfds with semi-ample K. If $\mathcal{X}_0$ is irreducible and with at most can. sing. Then $0 \in \Delta$ is at finite g_m distance $\forall m$ st g_m is defined.

We expect the converse is true.

Thm 4: Let $\mathcal{X}^\times \xrightarrow{\pi} \Delta^\times$ be a proj sm family of curves of genus $g \geq 2$. Then g_m is defined $\forall m \in \mathbb{N}$. The incompleteness of g_m for any 3 values of m's $\Rightarrow$ up to a base change, π can be completed into a smooth family.

Sketch of proof of Thm 1: V_s = bundle of $H^n(\mathcal{X}_s, \mathbb{C})_{pr}$.

Set up: polarized

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"] \\ \Delta^\times = S \end{tikzcd}$$

Hodge filtration $F: F^0 \supset F^1 \supset \cdots \supset F^n = \pi_* K_{\mathcal{X}/S}$

$$Q(u,v)|_{F_s} = \sqrt{-1}^{\,n} \int_{\mathcal{X}_s} u \cup v$$

$$\omega = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det \left[Q(u_i, \bar{u}_j) \right]_{i,j=1}^{p_g}$$

$\{u_i\}$ local frame of F^n

Period map: Fix ref's fiber V.

$$\begin{tikzcd} \phi : \Delta^\times \ni s \arrow[r] & \langle T \rangle \backslash D \\ \Phi : \mathbb{H} \ni z \arrow[u] \arrow[r] & D \arrow[u] \end{tikzcd}$$

$s = e^{2\pi\sqrt{-1} z}$

assume that T uni-potent

$N := \log T$ (or $I - T$).

upper half plane

$$A(z) := e^{-zN} \Phi(z) : \mathbb{H} \to \check{D} \quad \text{(cpt dual)}$$

$$\begin{tikzcd} \mathbb{H} \arrow[r] \arrow[d] & \check{D} \\ \Delta^\times \arrow[ur, "\exists \alpha"'] & \end{tikzcd}$$

$A(z+1) = A(z) \Rightarrow$

Schmid's Nilpotent Orbit Thm:

$\alpha(s)$ extends holomorphically over $s = 0$.

$F_\infty := \alpha(0)$ is the limiting Hodge filtration.

Near $s=0$, represent α^n via p_g vectors in V

$\vec{a}^j(s)$, holo in s — n-th flag

and $A_j(z) = \vec{a}^j(e^{2\pi\sqrt{-1} z}) = a_0^j + h$

Local frame $\Omega_j(z) := e^{zN} A_j(z)$, $1 \leq j \leq p_g$.

(1) the Kähler form of the g_H on $\mathbb{H}$ is

$$\omega = -\frac{\sqrt{-1}}{2}\,\partial\bar\partial \log \det\left[Q\left(e^{zN}A_i(z),\, e^{\bar z N}\overline{A_j(z)}\right)\right]_{i,j=1}^{n}$$

$$\det\left[Q\left(e^{zN}a_0^i,\, e^{\bar z N}\bar a_0^j\right) + h\right]$$

$$\det\left[Q\left(e^{2\sqrt{-1}\,yN}a_0^i,\, \bar a_0^j\right)\right] + h$$

$$p(y) + h$$

polynomial in y

here $z = x + iy$.

$e^{2\pi\sqrt{-1}\,z} = e^{2\pi\sqrt{-1}\,x}\cdot e^{-2\pi y}$ has property h: All partial derivatives $\to 0$ exponentially as $y \to \infty$, with decay rate indep of x.

Lemma: $d := \deg p(y) = 0 \iff N F_\infty^n = 0$.

This follows from the "polarization condition" on the Mixed Hodge str on $(V;\, F_\infty,\, W)$

monodromy weight filtration.

since $\dim \Delta^* = 1$:

$$g_H = G\,|dz|^2, \qquad G = -\frac{1}{4}\Delta \log \det Q$$

$$4G = \frac{(p'+h)^2 - (p+h)(p''+h)}{(p+h)^2} = \frac{(p'^2 - pp'') + h}{p^2 + h}$$

$$\sim \frac{p'^2 - pp''}{p^2} + h \sim \frac{d^2 - d(d-1)}{y^2} + h = \frac{d}{y^2} + h$$

(since $p^{-2}h \in h$)

If $NF_\infty^n = 0$, i.e. $d = 0$, then $G = h$

$\int_{z_0}^{\infty} \sqrt{G}\,|dz| < \infty$ for eg. the line $x = c$.

If $NF_\infty^n \neq 0$, i.e. $d \geq 1$, then $h < \frac{1}{y^3}$ for $y \gg 0$ unif in x

$$\int_{z_0}^{\infty} \sqrt{G}\,|dz| \gtrsim \sqrt{d}\,\log y\,\Big|_{y_0}^{\infty} = \infty$$

for any path with $y \mapsto \infty$.

(2) By the Clemens-Schmid exact sequence (or the invariant cycle thm), for $\pi: \mathcal{X} \to S$ s.s. $P_g(\mathcal{X}_t) \geq \sum P_g(X_i)$ with RHS corr. to T-inv cycles in F_s^n. hence "$=$" $\Leftrightarrow$ $NF_\infty^n = 0$.

(3) $\begin{matrix}\mathcal{X}\\ \downarrow \\ \Delta\end{matrix}$ $\mathcal{X}_0$ Gorenstein $\Rightarrow$ $\mathcal{X}$ Gorenstein

Also $K_{\mathcal{X}_s} = (K_{\mathcal{X}} + \mathcal{X}_s)|_{\mathcal{X}_s} = K_{\mathcal{X}}|_{\mathcal{X}_s}$

hence $h^0(\mathcal{X}_0, K_{\mathcal{X}_0}) \geq P_g$ by semi-continuity.

For $\mathcal{X}' \to \mathcal{X}$ a $\underset{\text{good}}{\text{resol}}$. then $\mathcal{X}_0' = \bigcup_{i=0}^N X_i'$

has one component $\phi: X_0' \to \mathcal{X}_0$ resol.

Then $h^0(X_0', K_{X_0'}) \geq h^0(\mathcal{X}_0, K_{\mathcal{X}_0}) \geq P_g$.

But then "$=$" holds and by (2) g_H is incomplete.

$\square$

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m-th quasi-Hodge metric g_m

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\phi"] \arrow[dr, "T"'] & \mathcal{X} \arrow[d, "\pi"] \\ & S \end{tikzcd}$$

m-cyclic cover along $D \in |K_{\mathcal{X}/S}^{\otimes m}|$

$\phi_* \mathcal{O}_{\mathcal{Y}} = \bigoplus_{k=0}^{m-1} K_{\mathcal{X}/S}^{-k}$

s.s. $\Rightarrow$ $T_* K_{\mathcal{Y}/S} = \bigoplus_{k=1}^{m} \pi_* K_{\mathcal{X}/S}^k$

$g_m := g_H$ wrt $\mathcal{Y}/S$. "sum of pluri canonical metrics"

Key: drawback: no monodromy action on each piece.

Q: for $S = \Delta^\times$,

g_m incomplete $\overset{?}{\longleftrightarrow}$ certain extension property of pluri-canonical forms

Proposal:

(1). For $\mathcal{X} \to \Delta$ semi-stable model, $\mathcal{X}_0 = \bigcup_{i=0}^N X_i$.

g_m incomplete $\Leftrightarrow$ $P_k(\mathcal{X}_s) = \sum_{i=0}^N \underset{\text{twisted}}{\tilde{P}_k}(X_i)$, $1 \leq k \leq m$

(2). (1) holds $\underset{\text{finite will be enough.}}{\underline{\forall m}}$ $\Rightarrow$ $\exists!$ X_0 with non-zero pluri-genera

(3) MMP

The same case holds (Thm 4).

sketch of pf:

$\mathcal{X} \to \Delta$ stable reduction

$\mathcal{X}_0 = \bigcup_{i \in I} X_i$, $g_i = g(X_i)$; $d_i = \sum_{j \neq i} X_j \cdot X_i = -X_i^2$ $\begin{cases} \geq 3 & g_i = 0 \\ \geq 1 & g_i \neq 0 \end{cases}$

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[dr] & \mathcal{X} \arrow[d] \\ & \Delta \end{tikzcd} \qquad P_g(\mathcal{Y}_s) = \sum_{k=1}^{m} P_k(\mathcal{X}_s)$$

Lemma: $P_g(Y_i) = \sum_{k=1}^{m} h^0(X_i, \tilde{K}_{X_i}^k)$

where $\tilde{K}_{X_i}^k := K_{X_i}^k \otimes \mathcal{O}_{X_i}(\sum_{j \neq i} X_j \cap X_i)^{k-1}$.

R.R. $\Rightarrow$ $h^0(\tilde{K}_{X_i}^k) - h^1(\tilde{K}_{X_i}^k) = k(2g_i - 2) + (k-1)d_i + (1 - g_i)$

$h^1(\tilde{K}_{X_i}^k) = 1$ for $k = 1$

$= 0$ for $k \geq 2$ (by stability)

so $\sum_{k=1}^{m} h^0(K_{\mathcal{X}_s}^k) = \sum_i \sum_{k=1}^{m} h^0(\tilde{K}_{X_i}^k)$

$1 + \sum_{k=1}^{m} (2k-1)(g-1) \qquad |I| + \sum_i \sum_{k=1}^{m} (2k-1)(g_i - 1) + (k-1)d_i$

i.e. $m^2(g - \sum_i g_i) = -(|I| - 1)m^2 + \frac{m(m-1)}{2} \sum d_i + (|I| - 1)$

if this holds for any 3 values of m then

$\Rightarrow$ $|I| = 1$, say $I = \{0\}$, and $g = \sum g_i = g(X_0)$

as expected. $\square$