Towards $A + B$ Theory in Conifold Transitions for Calabi–Yau Threefolds

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Contents

I. Calabi–Yau 3-folds

- A projective manifold $X/\mathbb{C}$ is Calabi–Yau if $\pi_1(X)$ is finite and $K_X = 0$ (or $c_1(X) = 0$).
- Yau's solution to the Calabi conjecture $\Longrightarrow$ for any cpt Kähler X with $c_1(X)_{\mathbb{R}} = 0$, $\exists$ finite cover $\widetilde{X} \to X$:

$$\widetilde{X} = A \times B \times C.$$

  $A \cong \mathbb{C}^g/\Lambda$ (flat), B is hyperkähler (SU(m)), and C is CY (SU(n)). Also $\pi_1(B) = \pi_1(C) = 0$, C is projective.
- The first new case appears in dim $= 3$. We have $h^1(\mathscr{O}) = h^2(\mathscr{O}) = 0$. WLOG we assume that $\pi_1(X) = 0$.
- **Question:** classification of CY 3-folds?
- What is the global structure (symmetries?) of $\mathscr{M}_{CY3}$?

- **Examples.** Adjunction formula for hypersurfaces $X \subset Y$:

$$K_X = (K_Y + X)|_X = 0 \Longleftrightarrow X \text{ is anti-canonical in Fano } Y.$$

- $X = (n+1) \subset P^n$. E.g. the Fermat hypersurfaces

$$x_0^{n+1} + \cdots + x_n^{n+1} = 0$$

is a CY $(n-1)$-fold. E.g. quintic 3-folds.

- $X = (\vec{d}_1, \ldots, \vec{d}_k) \subset \prod_{i=1}^m P^{n_i}$ with $\vec{d}_j = (d_{ji})_{i=1}^m$ and

$$\sum_{j=1}^k d_{ji} = n_i + 1, \qquad 1 \leq i \leq m.$$

This is a CICY of dimension $D = \sum n_i - m$.

- Let N_D be the numbers of them. Then $N_3 = 7890$.

- **Toric CY.** A lattice polytope $\triangle \subset M_{\mathbb{R}}$, $M \cong \mathbb{Z}^{n+1}$ is reflexive if $0 \in \operatorname{int} \triangle$ and its polar (dual) polytope

$$\triangle^\circ := \{w \in N := M^\vee \mid \langle w, v\rangle \geq -1,\ \forall v \in \triangle\}$$

is also a lattice polytope, in $N_{\mathbb{R}}$.

- Number of them [Kruezer–Skarke, 2000]:

$$\mathbf{N}_1 = 16, \quad \mathbf{N}_2 = 4319, \quad \mathbf{N}_3 = 473800776, \ldots.$$

- For a reflexive pair $(\triangle, \triangle^\circ)$, the toric variety

$$P_\triangle := \operatorname{Proj}\big(\bigoplus_{k\geq 0} \mathbb{C}^{k\triangle\cap M}\big)$$

is Fano with $H^0(K_{P_\triangle}^{-1}) = \bigoplus_{v\in\triangle\cap M} \mathbb{C}\, t^v$; similarly for $P_{\triangle^\circ}$.

- For a general section f, $X_f := \{f = 0\}$ is a CY n-fold.

II. Classical A model and B model

- The Hodge numbers of a CY 3-fold X are

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{22} & & 0 & \\ 1 & & h^{21} & & h^{12} & & 1 \\ & 0 & & h^{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

- $h^{11} = h^1(X, \Omega_X) = h^2$ parametrizes Kähler classes.
- $A(X) = QH(X)$ is the $g = 0$ Gromov–Witten theory in

$$\omega = B + iH \in \mathscr{K}_X^{\mathbb{C}} = H^2(X, \mathbb{R}) \oplus \sqrt{-1}\mathrm{Amp}(X).$$

- $h^{21} = h^1(X, T_X)$ parametrizes complex deformations.
- $B(X) = (\mathcal{H}^3, \nabla^{GM})$ is the VHS on the complex moduli $\mathscr{M}_X$ under the Gauss–Manin connection with lattice $H^3(X, \mathbb{Z})$.

- **A model.** For a CY 3-fold X, let $\beta \in H_2(X, \mathbb{Z})$,

  $\overline{M}(X, \beta) = \{h : C \to X \text{ stable} \mid C \text{ is nodal}, p_a(C) = 0, h_*[C] = \beta\} / \sim .$

- Virtual dim $= 0$: the essential genus 0 GW invariants are

$$n_\beta^X = \langle - \rangle_\beta^X = \int_{[\overline{M}(X,\beta)]^{virt}} \mathbf{1} \in \mathbb{Q}.$$

- *Toric example.* Let $X_f \subset P_\triangle$ with $f \in H^0(K_{P_\triangle}^{-1})$.
- $A(X_f)$ is *determined* by $\mathbb{C}^\times$**-localization** data [LLY, G 1999]:

$$I^X(q^\bullet, z^{-1}) = \sum_{\beta \in H_2(X_f, \mathbb{Z})} q^\beta \frac{\prod_{m=1}^{K^{-1}.\beta} (K_{P_\triangle}^{-1} + mz)}{\prod_{\rho \in \Sigma_1} \prod_{m=1}^{D_\rho.\beta} (D_\rho + mz)}.$$

- Σ is the *normal fan* of $P_\triangle$ and D_ρ is the *torus invariant divisor* corresponding to the one-edge $\rho \in \Sigma_1$, $q^\beta = e^{2\pi i(\beta.\omega)}$.

- **B model.** For the CY family $\pi : \mathscr{X} \to S := \mathscr{M}_X$,

$$\mathcal{H}^3 := R^3\pi_*\mathbb{C} \otimes \mathscr{O}_S \to S,$$

$F^p = \pi_*\Omega^p_{\mathscr{X}/S}$, $\mathcal{H}^{pq} = F^p \cap \overline{F^q}$, $\Omega \in \Gamma(S, F^3)$. Then

$$\nabla^{GM} F^p \hookrightarrow F^{p-1} \otimes \Omega^1_S, \qquad \langle \nabla^{GM}_{\partial/\partial x_j}\Omega \rangle_{j=1}^{h^{21}} = \mathcal{H}^{21}.$$

- **Periods.** Let $\delta_m \in H_3(X)$ be a basis with dual $\delta_m^* \in H^3(X)$. For $\eta \in \Gamma(S, \mathcal{H}^3)$, since $\nabla^{GM}\delta_m^* = 0$, we have

$$\nabla^{GM}_{\partial/\partial x_j}\eta = \sum_m \delta_m^* \frac{\partial}{\partial x_j} \int_{\delta_m} \eta, \qquad j \in [1, h^{21}].$$

GM$\Longleftrightarrow$ *Picard–Fuchs equations of period integrals* $\int_{\delta_m} \Omega$.

- *Toric example*: $B(X_f)$ is *determined* by the GKZ* system:

(1) symmetry operators;

(2) for ℓ a relation of $m_i \in \triangle \cap M$ with $\sum \ell_i = 0$,

$$\Box_\ell := \prod\nolimits_{\ell_i > 0} \partial_i^{\ell_i} - \prod\nolimits_{\ell_i < 0} \partial_i^{-\ell_i}.$$

III. Mirror, flops, and transitions

- **Mirror symmetry.**
- *Topological MS*: (Y, Y°) is a mirror pair of CY 3-folds if

$$h^{21}(Y) = h^{11}(Y^\circ), \qquad h^{11}(Y) = h^{21}(Y^\circ).$$

- *Classical MS*, or $A \leftrightarrow B$ MS: $B(Y) \cong A(Y^\circ), A(Y) \cong B(Y^\circ)$.
- *Toric Example*: Consider 2 families of CY 3-folds

$$X_f \subset P_\triangle, \qquad X_g^\circ \subset P_{\triangle^\circ}.$$

- Topological MS holds [Batyrev '94].
- $A \leftrightarrow B$ MS holds for "many cases".
- Observation: $\Sigma_1 =$ rays from 0 to $\mathrm{Vert}(\triangle^\circ)$.
- [HLY 1998] $\exists$ max-deg-point ($\Rightarrow$ mirror transform).

- **Flops.** A D-flop between CY 3-folds is a birational diagram

$$\begin{array}{ccc} \ell \subset Y & \overset{f}{\dashrightarrow} & Y' \supset \ell' \\ & {\scriptstyle \psi} \searrow \quad \swarrow {\scriptstyle \psi'} & \\ & \overline{X} & \end{array}$$

  where ψ is D-negative (log-extremal) and ψ' is D'-positive.
- [Kollár, Kawamata 1988] Birational CY 3-folds are connected by flops. 3D flops are classified.
- [Kollár–Mori 1992] Birational CY 3-folds Y and Y' have

$$\mathscr{M}_X \cong \mathscr{M}_{X'} \Longrightarrow B(Y) \cong B(Y')$$

  since flops can be *performed in flat families*.
- [Li–Ruan 2000] $A(Y) \cong A(Y')$ under $q^\beta \mapsto q^{f_*\beta}$ ($\ell \mapsto -\ell'$).

- **Transitions.**
- Geometric transition $X \nearrow Y$ (or $Y \searrow X$) of CY 3-folds:

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow \psi \\ X & \rightsquigarrow & \overline{X} \end{array} \qquad \begin{array}{l} K_Y = \psi^* K_{\overline{X}}, \\ \\ NF_\infty^3 = 0. \end{array}$$

- $X \nearrow Y$ is a conifold transition if $\overline{X}_{\text{sing}}$ has only ODPs

$$(\overline{X}, p_i) := \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\}.$$

- **Q1** [Reid 1987] Can ALL CY 3-folds be connected through (possibly non-projective) conifold transitions?
- **Q2** [W 2009] Does $(A(X), B(X))$ determines $(A(Y), B(Y))$ and vice versa? Notice $A(X) < A(Y)$ and $B(X) > B(Y)$.

IV. An observation from ordinary k-fold singularity

A + B Model in Quantum Geometry

Oct. 12, 2009

NTU Math

Dragon

LOCAL EXAMPLES: Consider the dim k hyper-surface $X_0 \subset C^{k+1}$:

$$x_0^k + x_1^k + \cdots + x_k^k = 0$$

with $p = 0 \in X_0$ being an ordinary k-fold singularity. The blow-up $f\colon X = Bl_p(X_0) \to X_0$ is crepant with exceptional divisor

$$E = (k) \subset P^k, \ N_{E/X} = O(-1)|_E.$$

The local structure of $E \subset X$, namely the germ (E, X) is equivalent to P^k "cut out" by the rank 2 vector bundle:

$$V_k = O(k) \oplus O(-1) \to P^k.$$

X_0 can be smoothed into a flat family $M \to \Delta$ with general smooth fiber $X' = M_t$. The semi-stable reduction $\pi\colon W \to \Delta$ is used to compare X and X' since $W_t = X'$ and $W_0 = X \cup E'$ for some Fano E'.

Quantum Transition from A to B:

The Gromov-Witten extremal function $f(a) = \Sigma_{d\in N} \langle a\rangle_{dL}\, q^{dL}$ attached to the extremal ray $L \in NE(X)$ can be calculated, using the quantum Serre duality principle, by the bundle

$$V_k^+ = O(k)\oplus O(1) \to P^k.$$

This is in turn reduced to $O(k) \to P^{k-1}$, the Calabi-Yau CY_k!

Where is the Picard-Fuchs operator P_k for $f(a)$?

Since $\dim CY_k = k - 2$, we must have $\deg P = k - 2$. But $\dim X' = k$. It must be the case that there is a "sub-VHS of $R^k\pi_*C$ of weight $k - 2$" which starts at $\Omega \in H^{n-1,1} = H^1(X', T)$. Let Γ be the vanishing cycle along π, then P_k is the Picard-Fuchs op for $\int_\Gamma \Omega$.

V. Statements for conifold transitions

Let $X \nearrow Y$ be a *projective* conifold transition of CY 3-folds through $\overline{X}$ with k ODPs $p_1, \ldots, p_k$, $\pi : \mathscr{X} \to \Delta$, $\psi : Y \to \overline{X}$:

$$\begin{array}{ccccc} & & C_i \subset Y & & N_{C_i/Y} = \mathscr{O}_{P^1}(-1)^{\oplus 2} \\ & & \big\downarrow \psi & & \\ N_{S_i/X} = T^*S^3 & S_i \subset X & \overset{\pi}{\rightsquigarrow} p_i \in \overline{X} & & \end{array}$$

Let $\mu := h^{2,1}(X) - h^{2,1}(Y) > 0$ and $\rho := h^{1,1}(Y) - h^{1,1}(X) > 0$.

$$\chi(X) - k\chi(S^3) = \chi(Y) - k\chi(S^2) \Longrightarrow \mu + \rho = k.$$

Hence there are non-trivial relations between the “vanishing cycles”:

$$A = (a_{ij}) \in M_{k\times\mu}, \qquad \sum_{i=1}^{k} a_{ij}[C_i] = 0,$$
$$B = (b_{ij}) \in M_{k\times\rho}, \qquad \sum_{i=1}^{k} b_{ij}[S_i] = 0.$$

Let $0 \to V_{\mathbb{Z}} \hookrightarrow H_3(X,\mathbb{Z}) \to H_3(\bar{X},\mathbb{Z}) \to 0$ and $V := C_{\mathbb{Z}} \otimes \mathbb{C}$.

Theorem (Basic exact sequence)

We have an exact sequence of weight two pure Hodge structures:

$$0 \to H^2(Y)/H^2(X) \xrightarrow{B} \mathbb{C}^k \xrightarrow{A^t} V \to 0.$$

Since $\psi : Y \to \overline{X}$ deforms in families, this identifies $\mathscr{M}_Y$ as a codimenison μ boundary strata in $\mathscr{M}_{\overline{X}}$ and *locally* $\mathscr{M}_{\overline{X}} \cong \Delta^\mu \times \mathscr{M}_Y$. Write $V = \mathbb{C}\langle \Gamma_1, \ldots, \Gamma_\mu \rangle$ in terms of a basis Γ_j's. Then the α-periods

$$r_j = \int_{\Gamma_j} \Omega, \qquad 1 \leq j \leq \mu$$

form the *degeneration coordinates* around $[\overline{X}]$. The discriminant loci of $\mathscr{M}_{\overline{X}}$ is described by a central hyperplane arrangement $D_B = \bigcup_{i=1}^k D_i$:

Proposition (Friedman 1986)

Let $w_i = a_{i1}r_1 + \cdots + a_{i\mu}r_\mu$, then the divisor $D_i := \{w_i = 0\} \subset \mathscr{M}_{\overline{X}}$ is the loci where the sphere S_i shrinks to an ODP p_i.

- The β-periods in transversal directions are given by a function u:

$$u_p = \partial_p u = \int_{\beta_p} \Omega$$

- The Bryant–Griffiths–Yukawa couplings extend over D_B and

$$u_{pmn} := \partial^3_{pmn} u = O(1) + \sum_{i=1}^{k} \frac{1}{2\pi\sqrt{-1}} \frac{a_{ip}a_{im}a_{in}}{w_i} = \int \partial_p \partial_m \partial_n \Omega \wedge \Omega$$

  for $1 \leq p, m, n \leq \mu$. It is holomorphic outside this index range.

- Let $y = \sum_{i=1}^k y_i e_i \in \mathbb{C}^k$, with e^i's being the dual basis on $(\mathbb{C}^k)^\vee$. The trivial logarithmic connection on $\underline{\mathbb{C}}^k \oplus (\underline{\mathbb{C}}^k)^\vee \longrightarrow \mathbb{C}^k$ is

$$\nabla^k = d + \frac{1}{z} \sum_{i=1}^k \frac{dy_i}{y_i} \otimes (e^i \otimes e_i^*).$$

Theorem (Local invariance: $\mathrm{Exc}(\mathscr{A}) + \mathrm{Exc}(\mathscr{B}) = $ trivial)

(1) *∇^k restricts to the logarithmic part of ∇^{GM} on V^*.*

(2) *∇^k restricts to the logarithmic part of $\nabla^{\mathrm{Dubrovin}}$ on $H^2(Y)/H^2(X)$.*

Theorem (Linked $\mathscr{A} + \mathscr{B}$ theory)

Let $[X]$ *be a nearby point of* $[\overline{X}]$ *in* $\mathscr{M}_{\overline{X}}$,

(1) $\mathscr{A}(X)$ *is a sub-theory of* $\mathscr{A}(Y)$ *(i.e. quantum sub-ring).*

(2) $\mathscr{B}(Y)$ *is a sub-theory of* $\mathscr{B}(X)$ *(sub-moduli, invariant sub-VHS).*

(3) $\mathscr{A}(Y)$ *can be reconstructed from a "refined* $\mathscr{A}$ *theory" on*

$$X^\circ := X \setminus \bigcup_{i=1}^{k} S_i$$

"linked" by the vanishing 3-spheres in $\mathscr{B}(X)$.

(4) $\mathscr{B}(X)$ *can be reconstructed from the variations of MHS on* $H^3(Y^\circ)$,

$$Y^\circ := Y \setminus \bigcup_{i=1}^{k} C_i,$$

"linked" by the exceptional curves in $\mathscr{A}(Y)$.

For (3) and (4), effective methods are under developed.

VI. Linked GW invariants for $\mathscr{A}$ model

- It is easy to see that $\mathscr{A}(X) \subset \mathscr{A}(Y)$: for $0 \neq \beta \in H_2(X)$, by degeneration formula in GW theory [Li] we can show

$$n_\beta^X = \sum_{\gamma \mapsto \beta} n_\gamma^Y.$$

- To determine $\mathscr{A}(Y)$ from $\mathscr{A}(X) + \mathscr{B}(X)$, it is equivalent to find a definition of each individual term n_γ^Y with the same β in terms of a refined data in X.

Lemma

$H_2(X^\circ) \cong H_2(Y^\circ) \cong H_2(Y)$. In particular, for a map $h : C \to X^\circ$, $\gamma := h_*[C] \in H_2(Y)$ is well-defined.

- This γ is called a linking data (β, L). It encodes the link between $h(C)$ (2D) and S_i's (3D) inside a 6D space X.

- If the stable maps $h : C \to X$ do not touch $\bigcup_i S_i^3$, then the linked GW invariants $n^X_{(\beta,L)}$ is defined and $n^X_{(\beta,L)} = n^Y_\gamma$.
- In general this is not true. However, it is true in the virtual sense, which is all we need:

Proposition

For X_t with $t \in \mathbb{A}^1 \setminus \{0\}$ small in the degenerating family $\pi : \mathscr{X} \to \mathbb{A}^1$ arising from the semi-stable reduction, we have a decomposition of the virtual class $[\overline{M}(X_t, \beta)]^{\mathrm{virt}}$ into a finite disjoint union of cycles

$$[\overline{M}(X_t, \beta)]^{\mathrm{virt}} = \coprod_{\gamma \in H_2(X^\circ)} [\overline{M}(X_t, \gamma)]^{\mathrm{virt}},$$

where $[\overline{M}(Y, \gamma)]^{\mathrm{virt}} \sim [\overline{M}(X_t, \gamma)]^{\mathrm{virt}} \in A_{\mathrm{vdim}}\left(\overline{M}(X_t, \beta)\right)$ is a cycle class corresponding to the linking data γ of X_t.

VII. Linked $\mathscr{B}$ model via VMHS

- $\mathscr{B}(Y)$ is a sub-theory of $\mathscr{B}(X)$ by viewing $\mathscr{M}_Y \hookrightarrow \mathscr{M}_{\overline{X}}$ as a boundary strata of $\mathscr{M}_X$.
- We will show that $\mathscr{B}(Y)$, together with the knowledge of extremal curves $Z := \bigcup_i C_i \subset Y$ determines $\mathscr{B}(X)$.

- Proposition

There is a short exact sequence of mixed Hodge structures

$$0 \to V \to H^3(X) \to H^3(U) \to 0, \tag{1}$$

where $H^3(X)$ is equipped with the limiting MHS of Schmid,

$$V \cong H_\infty^{1,1} H^3(X),$$

and $H^3(U)$ is equipped with the canonical mixed Hodge structure of Deligne. In particular, $F^3H^3(X) \cong F^3H^3(U)$, $F^2H^3(X) \cong F^2H^3(U)$.

- We have on $\overline{X}$, $U \cong \overline{X}^\circ := \overline{X} \setminus p$, where $p = \bigcup\{p_i\}$:

$$\cdots H^1_p(\Theta_{\bar{X}}) \to H^1(\Theta_{\bar{X}}) \to H^1(U, T_U) \to H^2_p(\Theta_{\bar{X}}) \to \cdots .$$

- [Schlessinge] p_i is a hypersurface singularity $\Longrightarrow$ depth $\mathscr{O}_{p_i} = 3 \Longrightarrow H^1_p(\Theta_{\bar{X}}) = 0$ and $H^2_p(\Theta_{\bar{X}}) \cong \bigoplus_{i=1}^k \mathbb{C}_{p_i}$:

$$0 \to H^1(\Theta_{\bar{X}}) \to H^1(U, T_U) \to H^2_p(\Theta_{\bar{X}}) \to \cdots .$$

- Comparing with the local to global spectral sequence

$$0 \to H^1(\Theta_{\bar{X}}) \xrightarrow{\lambda} Ext^1(\Omega_{\bar{X}}, \mathscr{O}_{\bar{X}}) \to H^0(\mathscr{E}xt^1(\Omega_{\bar{X}}, \mathscr{O}_{\bar{X}})) \xrightarrow{\kappa} H^2(\Theta_{\bar{X}}),$$

- $\Rightarrow \mathrm{Def}(\bar{X}) \cong H^1(U, T_U)$. Similarly, for $Y \supset Z = \bigcup C_i$ we get

$$\mathrm{Def}(Y) = H^1(T_Y) \subset H^1(U, T_U) \cong \mathrm{Def}(\bar{X}),$$

and then $\mathscr{M}_Y \hookrightarrow \mathscr{M}_{\bar{X}}$ (unobstructedness theorem).
- Write $\mathscr{I} := \mathscr{I}_{\mathscr{M}_Y}$ as the ideal sheaf of $\mathscr{M}_Y \subset \mathscr{M}_{\bar{X}}$.

- Since $H^2(U, T_U) \neq 0$, the deformation of U could be obstructed. Nevertheless, *the first-order deformation of U exists* and is parameterized by $H^1(U, T_U) \supset \mathrm{Def}(Y)$.
- Therefore, we have the following *smooth family*

$$\pi : \mathfrak{U} \to \mathcal{Z}_1 := Z_{\mathscr{M}_{\bar{X}}}(\mathscr{I}^2) \supset \mathscr{M}_Y,$$

where $\mathcal{Z}_1 = Z_{\mathcal{M}_{\bar{X}}}(\mathscr{I}^2)$ stands for the nonreduced subscheme of $\mathscr{M}_{\bar{X}}$ as the first jet extension of $\mathscr{M}_Y$ in $\mathscr{M}_{\bar{X}}$.
- [Katz] ∇^{GM} for $\pi : \mathfrak{U} \to \mathcal{Z}_1$ is defined by the lattice $H^3(U, \mathbb{Z}) \subset H^3(U, \mathbb{C})$. It underlies VMHS instead of VHS.
- The proposition implies

$$W_i H^3(U) = 0, \quad i \leq 2; \qquad W_3 \subset W_4$$

with $\mathrm{Gr}_3^W H^3(U) \cong H^3(Y)$ and $\mathrm{Gr}_4^W H^3(U) \cong V^*$.

- The Hodge filtration of the local system $F^0 = H^3(U, \mathbb{C})$:

$$F^\bullet = \{F^3 \subset F^2 \subset F^1 \subset F^0\}$$

  satisfies Griffiths' transversality.
- Since $K_U \cong \mathcal{O}_U$, F^3 is a line bundle over $\mathcal{Z}_1$ spanned by $\Omega \in \Omega^3_{\mathcal{U}/\mathcal{Z}_1}$. Near $[Y] \in \mathcal{Z}_1$,

$$F^2 \text{ is then spanned by } \Omega \text{ and } v(\Omega)$$

  where v runs through a basis of $H^1(U, T_U)$.
- Notice that $v(\Omega) \in W_3$ precisely when $v \in H^1(Y, T_Y)$.
- Proposition $\Rightarrow F^3 \subset F^2$ on $H^3(U)$ over $\mathcal{Z}_1$ lifts uniquely to $\tilde{F}^3 \subset \tilde{F}^2$ on $H^3(X)$ over $\mathcal{Z}_1$ with

$$\tilde{F}^3 \cong F^3, \qquad \tilde{F}^2 \cong F^2.$$

- The complete lifting $\tilde{F}^\bullet$ is then determined since

$$\tilde{F}^1 = (\tilde{F}^3)^\perp$$

by the first Hodge–Riemann relation on $H^3(X)$.

- Now $\tilde{F}^\bullet$ over $\mathcal{Z}_1$ uniquely determines a horizontal map

$$\mathcal{Z}_1 \to \check{\mathbb{D}}.$$

- Since it has maximal tangent dimension

$$h^1(U, T_U) = h^1(X, T_X),$$

it determines *the maximal horizontal slice*

$$\psi : \mathscr{M} \to \check{\mathbb{D}}$$

with $\mathscr{M} \cong \mathscr{M}_{\overline{X}}$ near $\mathscr{M}_Y$.

- Let Γ be the monodromy group generated by the local monodromy $T^{(i)} = \exp N^{(i)}$ around the divisor

$$D_i := \{w_i = \sum_{j=1}^{\mu} a_{ij} r_j = 0\}.$$

- Under the coordinates $\mathbf{t} = (r, s)$, the period map

$$\phi : \mathcal{M}_X = \mathcal{M}_{\bar{X}} \setminus \bigcup_{i=1}^{k} D^i \to \mathbb{D}/\Gamma$$

is then given (by an extension of Schmid's NOT) as

$$\phi(r,s) = \exp\left(\sum_{i=1}^{k} \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right) \psi(r,s),$$

- Since $N^{(i)}$ is determined by the Picard–Lefschetz formula, the period map ϕ is completely determined by A and C_i's.
- Hence the refined $\mathscr{B}$ model on $Y \setminus Z = U$ determines the $\mathscr{B}$ model on X.

END