Mean field equations, hyperelliptic curves, and modular forms I, II

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The 6th TIMS-OCAMI-WASEDA Workshop
March 25–26, 2014

LECTURE ONE

- Joint project with Chang-Shou Lin.
- The Green function $G(z, w)$ on a flat torus $E = E_\Lambda = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is the unique function on $E \times E$ which satisfies

$$-\triangle_z G(z, w) = \delta_w(z) - \frac{1}{|E|}$$

  and $\int_E G(z, w)\, dA = 0$, where δ_w is the Dirac measure with singularity at $z = w$.

- Because of the translation invariance of $\triangle_z$, we have $G(z, w) = G(z - w, 0)$ and it is enough to consider *the Green function* $G(z) := G(z, 0)$. Asymptotically

$$G(z) = -\frac{1}{2\pi} \log |z| + O(|z|^2).$$

- G can be explicitly solved in terms of elliptic functions.
- Let $z = x + iy$, $\tau := \omega_2/\omega_1 = a + ib \in \mathbb{H}$ and $q = e^{\pi i \tau}$ with $|q| = e^{-\pi b} < 1$. Then we denote $E = E_\tau$ and

$$\vartheta_1(z;\tau) := -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z}.$$

- (Neron): On E_τ (notice the τ dependence),

$$G(z;\tau) = -\frac{1}{2\pi} \log \left| \frac{\vartheta_1(z;\tau)}{\vartheta_1'(0;\tau)} \right| + \frac{1}{2b} y^2 + C(\tau).$$

- The structure of G, especially its critical points and critical values, will be the fundamental objects that interest us. $\nabla G(z) = 0$ on $E_\tau \iff$

$$\frac{\partial G}{\partial z} \equiv \frac{-1}{4\pi} \left((\log \vartheta_1)_z + 2\pi i \frac{y}{b} \right) = 0.$$

- Recall the Weierstrass elliptic functions wrt. Λ :

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big),$$
$$\zeta(z) = -\int^z \wp = \frac{1}{z} + \cdots,$$
$$\sigma(z) = \exp \int^z \zeta(w)\, dw = z + \cdots.$$

- σ is entire, odd with a simple zero on lattice points and

$$\sigma(z+\omega_i) = -e^{\eta_i(z+\frac{1}{2}\omega_i)}\sigma(z)$$

with $\eta_i = \zeta(z+\omega_i) - \zeta(z) = 2\zeta(\frac{1}{2}\omega_i)$ the quasi-periods.

- Indeed

$$\sigma(z) = e^{\eta_1 z^2/2} \frac{\vartheta_1(z)}{\vartheta_1'(0)}.$$

Hence $\zeta(z) - \eta_1 z = (\log \vartheta_1(z))_z$.

- We set $\omega_1 = 1, \omega_2 = \tau = a + bi, \omega_3 = \omega_1 + \omega_2$, and $E = E_\tau$. $z = t\omega_1 + s\omega_2 = (t + sa) + sbi = x + yi$.
- By Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$, we compute

$$
\begin{aligned}
(\log \vartheta_1)_z + 2\pi i \frac{y}{b} &= \zeta(z) - \eta_1 z + 2\pi i s \\
&= \zeta(z) - \eta_1 t - \eta_1 s\omega_2 + (\eta_1\omega_2 - \eta_2)s \\
&= \zeta(z) - t\eta_1 - s\eta_2.
\end{aligned}
$$

- Hence $\nabla G(z) = 0$ if and only if

$$
G_z = -\frac{1}{4\pi}\Big(\zeta(t\omega_1 + s\omega_2) - (t\eta_1 + s\eta_2)\Big) = 0.
$$

- **Question:** How many critical points can G have in E? What is the dependence in $\tau \in \mathbb{H}$?

- The 3 half periods are trivial critical points. Indeed,

$$G(z) = G(-z) \Rightarrow \nabla G(z) = -\nabla G(-z).$$

  Let $p = \frac{1}{2}\omega_i$ then $p = -p$ in E and so $\nabla G(p) = -\nabla G(p) = 0$.
- Other critical points must appear in pair $\pm z \in E$.

Example (Maximal principle)

For rectangular tori E: $(\omega_1, \omega_2) = (1, \tau = bi)$, $\frac{1}{2}\omega_i$, $i = 1, 2, 3$ are precisely all the critical points.

Example ($\mathbb{Z}_3$ symmetry)

For the torus E with $\tau = \rho := e^{\pi i/3}$, there are at least 5 critical points: 3 half periods $\frac{1}{2}\omega_i$ plus $\frac{1}{3}\omega_3$, $\frac{2}{3}\omega_3$.

- However, it is very difficult to study the critical points from the "simple equation" $\zeta(t\omega_1 + s\omega_2) = t\eta_1 + s\eta_2$ directly.

- **In PDE**, the geometry of $G(z,w)$ plays fundamental role in the non-linear mean field equations (= Liouville equation with singular RHS): On a flat torus E it takes the form ($\rho \in \mathbb{R}_+$)

$$\triangle u + e^u = \rho \delta_0.$$

- Originated from the prescribed curvature problem (Nirenberg problem, constant $K = 1$ with cone metrics etc.).
- The mean field limit of Euler flow in statistic physics. Related to the self-dual condensation of abelian Chern-Simons-Higgs model (Nolasco and Tarantello 1999).
- **In Arithmetic Geometry**, $G(z,w)$ also appears in the Arakelov geometry as the intersection number of two sections z and w of the arithmetic surface $\mathcal{E} \to \operatorname{Spec} \mathbb{Z} \cup \{\infty\}$ at the ∞ fiber $\mathcal{E}_\infty =$ Riemann surface E.

- When $\rho \notin 8\pi\mathbb{N}$, it has been proved by C.-C. Chen and C.-S. Lin that the Leray-Schauder degree is

$$d_\rho = n + 1 \quad \text{for} \quad \rho \in (8n\pi, 8(n+1)\pi),$$

so the equation has solutions, independent of the shape of E.

- The first interesting case is when $\rho = 8\pi$ where the degree theory fails completely.

Theorem (Existence criterion via ∇G for $n = 1$)

For $\rho = 8\pi$, the mean field equation on a flat torus $E = \mathbb{C}/\Lambda$:

$$\triangle u + e^u = 8\pi\delta_0$$

has solutions if and only if the G has more than 3 critical points. Moreover, each extra pair of critical points $\pm p$ corresponds to an one parameter family of solutions u_λ, where $\lim_{\lambda\to\infty} u_\lambda(z)$ blows up precisely at $z \equiv \pm p$.

- **Structure of solutions.**
- Liouville's theorem says that any solution u of $\triangle u + e^u = 0$ in a simply connected domain $\Omega \subset \mathbb{C}$ must be of the form

$$u = \log \frac{8|f'|^2}{(1+|f|^2)^2},$$

where f, called a developing map of u, is meromorphic in Ω.

- It is straightforward to show that for $\rho = 8\pi\eta \in \mathbb{R}$,

$$S(f) \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = u_{zz} - \frac{1}{2}u_z^2 = -2\eta(\eta+1)\frac{1}{z^2} + O(1).$$

I.e., any developing map f of u has the same Schwartz derivative $S(f)$, which is elliptic on E. Hence $S(f) = -2(\eta(\eta+1)\wp(z) + B)$.

- By the theory of ODE, locally $f = w_1/w_2$ for two solutions w_i of the Lamé equation $L_{\eta,B}\, y = 0$:

$$y'' + \frac{1}{2}S(f)y = y'' - (\eta(\eta+1)\wp(z) + B)y = 0$$

  for some $B \in \mathbb{C}$.

- Furthermore, for any two developing maps f and $\tilde{f}$ of u, there exists $S = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \in PSU(2)$ such that $\tilde{f} = Sf := \dfrac{pf - \bar{q}}{qf + \bar{p}}$.
- So, solutions to the mean field equation correspond to Lamé equations with **unitary projective monodromy groups**.

- Geometrically the Liouville equation is simply the prescribing Gauss curvature equation in the new metric $g = e^w g_0$ over D, where $w = u/2 - \log\sqrt{2}$ and g_0 is the Euclidean flat metric on $\mathbb{C}$:

$$K_g = -e^{-u}\triangle u = 1. \tag{1}$$

- It is then clear the inverse stereographic projection $\mathbb{C} \to S^2$

$$(X, Y, Z) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}\right)$$

provides solutions to (1) with conformal factor

$$e^w = e^{\frac{1}{2}u - \frac{1}{2}\log 2} = \frac{2}{1+|z|^2}.$$

- Starting from this special solution for $D = \Delta$, the unit disk, general solutions on simply connected domain D can be obtained by using the Riemann mapping theorem via a holomorphic map
$$f : D \to \Delta.$$
- The conformal factor is then the one as expected:
$$e^u = \frac{8|f'|^2}{(1+|f|^2)^2}.$$
- The problem is to glue the local developing maps to a "global one". This is a monodromy problem on the once punctured torus $E^\times = E\backslash\{0\}$. Since it is homotopic to "8", we have
$$\pi_1(E^\times, x_0) = \mathbb{Z} * \mathbb{Z}$$
being a free group of rank two.

Lemma (Developing map for $\eta = \frac{1}{2}\ell \in \frac{1}{2}\mathbb{Z}$)

Given Λ, for $\rho = 4\pi\ell$, $\ell \in \mathbb{N}$, by analytic continuation across Λ, f is glued into a meromorphic function on $\mathbb{C}$. (Instead of on $E = \mathbb{C}/\Lambda$.)

- **First constraint from the double periodicity:**

$$f(z+\omega_1) = S_1 f, \quad f(z+\omega_2) = S_2 f$$

  with $S_1 S_2 = \pm S_2 S_1$ (abelian projective monodromy).

- **Second constraint from the Dirac singularity:**
  1. If $f(z)$ has a zero/pole at $z_0 \notin \Lambda$ then order $r = 1$.
  2. $f(z) = a_0 + a_{\ell+1}(z - z_0)^{\ell+1} + \cdots$ be regular at $z_0 \in \Lambda$.

- **Type I (Topological) Solutions** $\Longleftrightarrow \ell = 2n+1$:

$$f(z+\omega_1) = -f(z), \qquad f(z+\omega_2) = \frac{1}{f(z)}.$$

Then $g = (\log f)' = f'/f$ takes the form

$$g(z) = \sum_{i=1}^{l} (\zeta(z-p_i) - \zeta(z-p_i-\omega_2)) + c$$

which is elliptic on $E' = \mathbb{C}/\Lambda'$, $\Lambda' = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ with the only (highest order) zeros at $z_0 \equiv 0 \pmod{\Lambda}$ of order $\ell = 2n+1$.

- The equations $0 = g(0) = g''(0) = g^{(4)}(0) = \cdots$ implies that f is an even function (a non-trivial symmetric function argument). So f has simple zeros at $\pm p_1, \ldots, \pm p_n$ and $\omega_1/2$.

- The remaining equations $0 = g'(0) = g'''(0) = g^{(5)}(0) = \cdots$ leads to the polynomial system for $\wp(p_i)$'s:

Theorem (Type I integrability, $\rho = 4\pi(2n+1)$)

(1) *For $\rho = 4\pi\ell$, $\ell = 2n+1$. All solutions are of type I and even. f has simple zeros at $\omega_1/2$ and $\pm p_i$ for $i = 1, \ldots, n$, and poles $q_i = p_i + \omega_2$.*

(2) *For $x_i := \wp(p_i)$, $\tilde{x}_i := \wp(q_i)$, and $m = 1, \ldots, n$,*

$$\sum_{i=1}^n x_i^m - \sum_{i=1}^n \tilde{x}_i^m = c_m, \quad (x_m - e_2)(\tilde{x}_m - e_2) = \mu,$$

for some constants c_m and $\mu = (e_2 - e_1)(e_2 - e_3)$. This is a $2n$ affine polynomial system in $\mathbb{C}^{2n}$ of degree $2^n n!$.

(3) *The corresponding Lamé equation $L_{\eta=n+1/2,B}\, y = 0$ has finite monodromy group M (in fact $PM = V_4$) hence there is a polynomial p_n of degree $n+1$ such that $p_n(B) = 0$. (Brioschi-Halphen 1894.)*

This is a far more precise than the degree counting formula.

Example ($\rho = 4\pi$, $n = 0$, unique solution)

The developing map is given by $f(z) = f(0) \exp \int_0^z g(w)\,dw$ with

$$g(z) = A \frac{\sigma_{E'}(z)\sigma_{E'}(z - \omega_2)}{\sigma_{E'}(z + \frac{1}{2}\omega_1)\sigma_{E'}(z - \frac{1}{2}\omega_1 - \omega_2)},$$

where the residue at $z = \frac{1}{2}\omega_1$ is fiexed to be 1 by choosing

$$A = \frac{\sigma_{E'}(\omega_1 + \omega_2)}{\sigma_{E'}(\frac{1}{2}\omega_1)\sigma_{E'}(\frac{1}{2}\omega_1 + \omega_2)}.$$

Here all elliptic functions are with respect to $\Lambda' = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$.

Example ($\rho = 12\pi$, $n = 1$, two solutions)

Let $p_1 = a$. $p_2 = -a$ and $p_3 = \frac{1}{2}\omega_1$. Then

$$(\wp(a) - e_2)^2 + \tfrac{1}{2}(e_1 - e_3)(\wp(a) - e_2) - \mu = 0.$$

The two solutions coincide precisely when $\tau = e^{\pi i/3}$.

- **Type II (Scaling Family) Solutions $\Longleftrightarrow \eta = n$ ($\ell = 2n$):**

$$f(z+\omega_1) = e^{2i\theta_1} f(z), \qquad f(z+\omega_2) = e^{2i\theta_2} f(z).$$

- If f satisfies this, $e^{\lambda} f$ also satisfies this for any $\lambda \in \mathbb{R}$. Thus

$$u_\lambda(z) = \log \frac{8e^{2\lambda}|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}$$

  is a scaling family of solutions with developing maps $\{e^{\lambda} f\}$.

- u_λ is a **blow-up sequence**. The blow-up points for $\lambda \to \infty$ (resp. $-\infty$) are precisely zeros (resp. poles) of $f(z)$.
- $g = (\log f)'$ is elliptic on $E = \mathbb{C}/\Lambda$, with highest order zero at $z = 0$ of order $\ell = 2n$.

- $0 = g'(0) = g'''(0) = \cdots = g^{(2n-1)}(0)$ implies that g is even.
- Suppose that $g(z)$ has zeros $\pm p_1, \cdots, \pm p_n$. We may write

$$g(z) = \frac{\wp'(p_1)}{\wp(z) - \wp(p_1)} + \cdots + \frac{\wp'(p_n)}{\wp(z) - \wp(p_n)}$$

  constraint by $0 = g''(0) = \cdots = g^{(2n-2)}(0)$. These give rise to the first $n-1$ equations on $p_1, \ldots, p_n$. ($g(0) = 0$ is automatic.)
- **To be written down and discussed in the next lecture**.
- And then

$$f(z) = f(0) \exp \int_0^z g(\xi)\, d\xi$$

  which should satisfies (the n-th equation)

$$\int_{L_i} g \in \sqrt{-1}\mathbb{R}, \qquad i = 1, 2.$$

- **Periods integrals.** Let L_1, L_2 be the fundamental 1-cycles. Set

$$F_i(p) := \int_{L_i} \Omega(\xi, p)\, d\xi,$$

where $p \not\equiv \frac{1}{2}\omega_i \pmod{\Lambda}$ and

$$\Omega(\xi, p) = \frac{\wp'(p)}{\wp(\xi) - \wp(p)} = 2\zeta(p) - \zeta(p+\xi) - \zeta(p-\xi).$$

Lemma (Periods integrals and critical points)

Let $p = t\omega_1 + s\omega_2$, then (up to $4\pi i\mathbb{N}$)

$$F_1(p) = 2(\omega_1\zeta(p) - \eta_1 p) = 2(\zeta(p) - t\eta_1 - s\eta_2)\omega_1 - 4\pi i s,$$
$$F_2(p) = 2(\omega_2\zeta(p) - \eta_2 p) = 2(\zeta(p) - t\eta_1 - s\eta_2)\omega_2 + 4\pi i t.$$

- Hence solution $\{u_\lambda\}$ corresponds to $\pm p \notin E[2]$ with $\nabla G(p) = 0$.

- When $\rho = 8\pi$ ($\ell = 2$), $p_1 = p$, $p_2 = -p$, $g(z) = \Omega(z, p)$ and

$$f(z) = f(0) \exp \int_0^z g(\xi)\, d\xi$$

  gives rise to a solution $\Longleftrightarrow$
  $F_i(p) \in \sqrt{-1}\,\mathbb{R}$, $i = 1, 2$, $\Longleftrightarrow \nabla G(p) = 0$.

- Theorem (Uniqueness, Lin–W 2006, 2010)

  For $\rho = 8\pi$, the mean field equation $\triangle u + e^u = \rho\delta_0$ on a flat torus has at most one solution **up to scaling**.

- Theorem (Number of critical points)

  The Green function has either 3 or 5 critical points.

  - We were unable to prove it from the critical point equation.

- Our proof on uniqueness is based on the method of symmetrization applied to the linearized equation at the unique **even solution** in u_λ (choose $\lambda = -\log|f(0)|$ to get $f(0) = 1$).
- In fact we prove uniqueness of the one parameter family

$$\triangle u + e^u = \rho\delta_0, \quad \rho \in [4\pi, 8\pi]$$

on E within *even solutions*, by the continuity method.

- **Theorem**

For $\rho \in [4\pi, 8\pi]$, Let u be a solution of $\triangle u + e^u = \rho\delta_0$, $u(-z) = u(z)$ in E (so $\int_E e^u = \rho$.) Then the linearized equation at u:

$$\begin{cases} \triangle\varphi + e^u\varphi = 0 \\ \varphi(z) = \varphi(-z) \end{cases} \quad \text{on } E$$

is non-degenerate, i.e. it has only trivial solution $\varphi \equiv 0$.

Sketch of the main idea:

Use $x = \wp(z)$ as two-fold covering map $E \to S^2 = \mathbb{C} \cup \{\infty\}$ and require $\wp$ being an isometry:

$$e^{u(z)}|dz|^2 = e^{v(x)}|dx|^2 = e^{v(x)}|\wp'(z)|^2|dz|^2.$$

Namely we set

$$v(x) := u(z) - \log|\wp'(z)|^2 \quad \text{and} \quad \psi(x) := \varphi(z).$$

There are four branch points on $\mathbb{C} \cup \{\infty\}$, $p_0 = \wp(0) = \infty$ and $p_j = e_j := \wp(\omega_j/2)$ for $j = 1, 2, 3$. Since $\wp'(z)^2 = 4\prod_{j=1}^{3}(x - e_j)$, then

$$\begin{cases} \triangle v + e^v = \sum_{j=1}^{3}(-2\pi)\delta_{p_j} \\ \triangle \psi + e^v \psi = 0 \end{cases} \quad \text{in } \mathbb{R}^2$$

At infinity let $y = 1/x$. The isometry reads as

$$e^{u(z)}|dz|^2 = e^{w(y)}|dy|^2 = e^{w(y)} \frac{|\wp'(z)|^2}{|\wp(z)|^4}|dz|^2,$$

$$w(y) = u(z) - \log \frac{|\wp'(z)|^2}{|\wp(z)|^4} \sim \left(\frac{\rho}{2\pi} - 2\right) \frac{1}{2} \log |y|.$$

Thus $\rho \geq 4\pi$ implies that p_0 is a singularity with non-negative α_0.

The total measure on E and $\mathbb{R}^2$ are then given by

$$\int_E e^u \, dz = \rho \leq 8\pi \quad \text{and} \quad \int_{\mathbb{R}^2} e^v \, dx = \frac{\rho}{2} \leq 4\pi.$$

The proof is then reduced to:

Theorem (Symmetrization lemma)

Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain and let v be a solution of

$$\triangle v + e^v = \sum_{j=1}^N 2\pi\alpha_j \delta_{p_j}$$

in Ω. Suppose that $\lambda_1 = 0$ for $\triangle + e^v$ on Ω with φ the first eigenfunction.
(i) If the isoperimetric inequality with respect to $ds^2 = e^v|dx|^2$:

$$2l^2(\partial\omega) \geq m(\omega)(4\pi - m(\omega))$$

holds for all level domains $\omega = \{\varphi \geq t\}$ with $t \geq 0$, then

$$\int_\Omega e^v \, dx \geq 2\pi.$$

(ii) Moreover, the isoperimetric inequality holds if there is only one negative α_j and $\alpha_j = -1$.

- It remains to study the geometry of critical points over $\mathcal{M}_1$, which relies on methods of deformations and the degeneracy analysis of half periods.

Theorem (Moduli dependence, Lin–W 2013)

(1) *Let $\Omega_3 \subset \mathcal{M}_1 \cup \{\infty\} \cong S^2$ (resp. Ω_5) be the set of tori with 3 (resp. 5) critical points, then $\Omega_3 \cup \{\infty\}$ is closed containing $i\mathbb{R}$, Ω_5 is open containing the vertical line $[e^{\pi i/3}, i\infty)$.*

(2) *Both Ω_3 and Ω_5 are simply connected with $C := \partial\Omega_3 = \partial\Omega_5$ homeomorphic to S^1 containing ∞.*

(3) *Moreover, the extra critical points are split out from some half period point when the tori move from Ω_3 to Ω_5 across C.*

(4) *(Strong uniqueness) The map $\Omega_5 \to [0,1]^2$ by $\tau \mapsto (t,s)$ for $p(\tau) = t\omega_1 + s\omega_2$ is a bijection onto $\triangle = [(\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2})]$.*

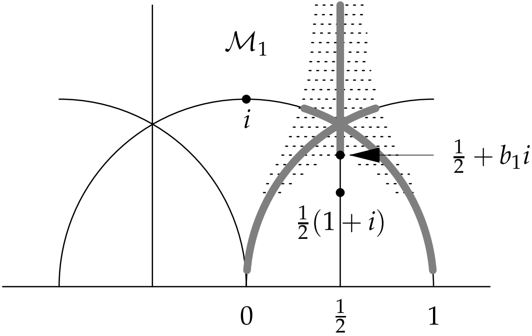

Figure : Ω_5 contains a neighborhood of $e^{\pi i/3}$.

- On the line $\operatorname{Re}\tau = 1/2$ which are equivalent to the rhombuses tori, the proof relies on *functional equations* of ϑ_1.
- The general case uses *modular forms of weight one*.

- **Idea of proof:**

$$\Psi(N) := \#\{\, (k_1, k_2) \mid (N, k_1, k_2) = 1, 0 \le k_i \le N-1 \,\}.$$

Consider the weight one modular function for $\Gamma(N)$:

$$\begin{aligned} Z_{N,k_1,k_2}(\tau) &:= \zeta\left(\frac{k_1\omega_1 + k_2\omega_2}{N}; \tau\right) - \frac{k_1\eta_1 + k_2\eta_2}{N} \\ &= -Z_{N,N-k_1,N-k_2}(\tau) \end{aligned}$$

(first studied by Hecke (1926));

- and the weight $\Psi(N)$ one for full modular group:

$$Z_N(\tau) := \prod_{(N,k_1,k_2)=1} Z_{N,k_1,k_2}(\tau) \in M_{\Psi(N)}(\mathrm{SL}(2,\mathbb{Z})).$$

- Each $\tau \in \mathbb{H}$ with $Z_N(\tau) = 0$ is (at least) a double zero.

- For odd $N \geq 5$, $\nu_i(Z_N) = \nu_\rho(Z_N) = 0$,
- At ∞, Hecke calculated the asymptotic expansion: $\nu_\infty(Z_N) = \phi(N/2) = 0$,
- Then the degree formula for modular forms (Riemann–Roch):

$$(Z_N)_{\text{red}} = \frac{1}{2}\deg Z_N = \frac{1}{2}\sum_p \nu_p(Z_N) = \frac{\Psi(N)}{24}.$$

- Take N prime, this suggests a 1-1 correspondence between Ω_5 and

$$\triangle = [(\tfrac{1}{3}, \tfrac{1}{3}), (\tfrac{1}{2}, \tfrac{1}{2}), (0, \tfrac{1}{2})]$$

under the map $\Omega_5 \to [0,1] \times [0, \frac{1}{2}]$:

$$\tau \mapsto (t, s), \quad \text{where} \quad p(\tau) = t\omega_1 + s\omega_2.$$

- **The actual proof:** Deformations in $t, s \notin \frac{1}{2}\mathbb{Z}$.
- Let $F \subset \mathbb{H}$ be the fundamental domain for $\Gamma_0(2)$ defined by

$$F := \{\tau \in \mathbb{H} \mid 0 \leq \operatorname{Re}\tau \leq 1, |\tau - \tfrac{1}{2}| \geq \tfrac{1}{2}\}.$$

We analyze solutions $\tau \in F$ for $Z_{t,s}(\tau) = 0$ by varying (t, s).

- For $\tau \in \partial F$, E is a rectangle and the only critical points of G are half periods. So $Z_{t,s}(\tau) \neq 0$ for $\tau \in \partial F$.
- Based on this, we use of the argument principle along the curve ∂F to analyze the number of zeros of $Z_{t,s}$ in F.
- We deduce from the Jacobi triple product formula that

$$Z_{t,s}(\tau) = 2\pi i(s - \tfrac{1}{2}) - \pi i \frac{2e^{2\pi i z}}{1 - e^{2\pi i z}} - 2\pi i \sum_{n=1}^{\infty} \left(\frac{e^{2\pi i z} q^n}{1 - e^{2\pi i z} q^n} - \frac{e^{-2\pi i z} q^n}{1 - e^{-2\pi i z} q^n} \right),$$

where $z = t + s\tau$.

- Lemma (Asymptotic behavior of $Z_{t,s}$ on cusps)

  We have $Z_{t,s}(-1/\tau) = \tau Z_{-s,t}(\tau)$*, and for* $t \in (0,1)$*,*

$$Z_{t,s}(\tau) = \frac{-1}{\tau} Z_{-s,t}(-1/\tau) = \frac{2\pi i}{\tau}\left(\tfrac{1}{2} - t + o(1)\right)$$

  as $\tau \to 0$*. Similarly,* $Z_{t,s}(\tau+1) = Z_{t+s,t}(\tau)$*, and for* $t+s \in (0,1)$*,*

$$Z_{t,s}(\tau) = Z_{t+s,s}(\tau-1) = \frac{2\pi i}{\tau-1}\left(\tfrac{1}{2} - (t+s) + o(1)\right).$$

- Lemma (Non-Vanishing)

  For any $\tau \in \mathbb{H}$*, the addition law implies that*

  (i) $\zeta\left(\frac{3}{4}\omega_1 + \frac{1}{4}\omega_2\right) \neq \frac{3}{4}\eta_1 + \frac{1}{4}\eta_2.$

  (ii) $\zeta\left(\frac{1}{6}\omega_1 + \frac{1}{6}\omega_2\right) \neq \frac{1}{6}\eta_1 + \frac{1}{6}\eta_2.$

- For (ii), we choose $z = \frac{1}{6}(\omega_1 + \omega_2) = \frac{1}{6}\omega_3$ and $u = \frac{1}{3}\omega_3$. Then

$$0 \neq \frac{\wp'(z)}{\wp(z) - \wp(u)} = \zeta(\tfrac{1}{2}\omega_3) + \zeta(-\tfrac{1}{6}\omega_3) - 2\zeta(\tfrac{1}{6}\omega_3)$$
$$= -3(\zeta(\tfrac{1}{6}\omega_1 + \tfrac{1}{6}\omega_2) - \tfrac{1}{6}\eta_1 - \tfrac{1}{6}\eta_2).$$

- Suppose that $(t,s) \in [0,1] \times [0,\frac{1}{2}] \backslash \{(0,0),(\frac{1}{2},0),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2})\}$. Then $Z_{t,s}(\tau) = 0$ has a solution $\tau \in \mathbb{H}$ if and only if that

$$(t,s) \in \triangle := \{(t,s) \mid 0 < t,s < \tfrac{1}{2},\, t+s > \tfrac{1}{2}\}.$$

  Moreover, the solution $\tau \in F$ is unique for any $(t,s) \in \triangle$.

- *Proof*: The cases $(t,s) \notin \triangle$ are excluded by the Lammas. From

$$\nu_\infty(Z_3) + \frac{1}{2}\nu_i(Z_3) + \frac{1}{3}\nu_\rho(Z_3) + \sum_{p \neq \infty, i, \rho} \nu_p(Z_3) = \frac{8}{12},$$

  $Z_{\frac{1}{3},\frac{1}{3}}(\rho) = Z_{\frac{2}{3},\frac{2}{3}}(\rho) = 0 \Longrightarrow \nu_\rho(Z_{(3)}) = 2$ and other terms $= 0$.
  Thus $\tau = \rho$ is a simple root to $Z_{\frac{1}{3},\frac{1}{3}}(\tau) = 0$. **QED**

Theorem (Periods integrals and type II solutions)

Consider the mean field equation $\triangle u + e^u = \rho\delta_0$ *on* $E = \mathbb{C}/\Lambda$.

- *If solutions exist for* $\rho = 8n\pi$*, then there is a unique even solution within each type II scaling family. (*$\ell = 2n$, $a_{n+i} = -a_i$.*)*
- *The solution* u *is determined by the zeros* $a_1, \ldots, a_n$ *of* f*. In fact*

$$g(z) = \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)}, \qquad f(z) = f(0) \exp \int^z g(\xi)\, d\xi.$$

- $\mathrm{ord}_{z=0}\, g(z) = 2n$ *leads to* $n-1$ *equations for* $a = \{a_1, \ldots, a_n\}$.
- *The* n*-th equation is given by* $\int_{L_i} g \in \sqrt{-1}\mathbb{R}$*, which is equivalent to*

$$\sum_{i=1}^{n} \nabla G(a_i) = 0.$$

- **The $n-1$ algebraic equations:**
- Under the notations $(w, x_j, y_j) = (\wp(z), \wp(p_j), \wp'(p_j))$,

$$g(z) = \sum_{j=1}^{n} \frac{1}{w} \frac{y_j}{1 - x_j/w}$$
$$= \sum_{j=1}^{n} \frac{y_j}{w} + \sum_{j=1}^{n} \frac{y_j x_j}{w^2} + \cdots + \sum_{j=1}^{n} \frac{y_j x_j^r}{w^{r+1}} + \cdots .$$

- Since $g(z)$ has a zero at $z = 0$ of order $2n$ and $1/w$ has a zero at $z = 0$ of order two, we get

$$\sum_{j=1}^{n} y_j x_j^r = \sum_{j=1}^{n} \wp'(a_j)\wp(a_j)^r = 0, \quad 0 \le r \le n-2.$$

Theorem (Green/polynomial system)

For $\rho = 8n\pi$, $n \in \mathbb{N}$, the n equations for $a = \{a_1, \ldots, a_n\}$ are precisely

$$\wp'(a_1)\wp^r(a_1) + \cdots + \wp'(a_n)\wp^r(a_n) = 0,$$

where $r = 0, \ldots, n-2$, and $\nabla G(a_1) + \cdots + \nabla G(a_n) = 0$.

Theorem (Hyperelliptic geometry/Lamé curve)

For $x_i := \wp(a_i)$, $y_i := \wp'(a_i)$, the first $n-1$ algebraic equations

$$\sum y_i x_i^r = 0, \quad r = 0, \ldots, n-2,$$

defines an affine hyperelliptic curve under the 2 to 1 map $a \mapsto \sum \wp(a_i)$:

$$X_n := \{(x_i, y_i)\} \subset \mathrm{Sym}^n E \longrightarrow (x_1 + \cdots + x_n) \in \mathbb{P}^1.$$

- **The proof relies on its relation to Lamé equations:**

$$f = \exp \int g\, dz = \exp \int \sum_{i=1}^{n} (2\zeta(a_i) - \zeta(a_i - z) - \zeta(a_i + z))\, dz$$

$$= e^{2\sum_{i=1}^{n} \zeta(a_i) z} \prod_{i=1}^{n} \frac{\sigma(z - a_i)}{\sigma(z + a_i)} = (-1)^n \frac{w_a}{w_{-a}},$$

where $w_a(z) := e^{z \sum \zeta(a_i)} \prod_{i=1}^{n} \frac{\sigma(z - a_i)}{\sigma(z)\sigma(a_i)}$ is **the basic element**.

- Theorem (Explicit map $a \mapsto B_a = (2n-1) \sum \wp(a_i)$)

$a \in X_n$ if and only if w_a and w_{-a} are **two** *solutions of the Lamé equation*

$$\frac{d^2 w}{dz^2} - \Big(n(n+1)\wp(z) + (2n-1) \textstyle\sum_{i=1}^{n} \wp(a_i)\Big) w = 0.$$

- This is a long calculation via the polynomial system (omitted).

- **Idea of proof on the hyperelliptic structure on X_n.**
- Consider $y^2 = p(x) = 4x^3 - g_2 x - g_3$, where $(x, y) = (\wp(z), \wp'(z))$, and we set $(x_i, y_i) = (\wp(a_i), \wp'(a_i))$. Consider a basis of solutions to the Lamé equation

  $$w'' = (n(n+1)\wp(z) + B)w$$

  (for some B) given by $w_a(z)$ and $w_{-a}(z)$.
- Let $X(z) = w_a(z)w_{-a}(z)$. By the addition theorem,

  $$X(z) = (-1)^n \prod_{i=1}^{n} \frac{\sigma(z+a_i)\sigma(z-a_i)}{\sigma(z)^2\sigma(a_i)^2} = (-1)^n \prod_{i=1}^{n} (\wp(z) - \wp(a_i)).$$

  That is, $X(x) = (-1)^n \prod_{i=1}^{n}(x - x_i)$ is a polynomial in x.

- **Key:** $X(z)$ satisfies the second symmetric power of the Lamé equation:

$$\frac{d^3X}{dz^3} - 4(n(n+1)\wp + B)\frac{dX}{dz} - 2n(n+1)\wp' X = 0.$$

- Hence $X(x)$ is a polynomial solution, in variable x, to

$$p(x)X''' + \frac{3}{2}p'(x)X'' - 4((n^2+n-3)x + B)X' - 2n(n+1)X = 0.$$

- X is determined by B and certain initial conditions.

- Write $X(x) = (-1)^n(x^n - s_1x^{n-1} + \cdots + (-1)^n s_n)$, this translates to a linear recursive relation for $\mu = 0, \cdots, n-1$:

$$\begin{aligned} 0 = 2(n-\mu)(2\mu+1)(n+\mu+1)s_{n-\mu} \\ - 4(\mu+1)Bs_{n-\mu-1} \\ + \tfrac{1}{2}g_2(\mu+1)(\mu+2)(2\mu+3)s_{n-\mu-2} \\ - g_3(\mu+1)(\mu+2)(\mu+3)s_{n-\mu-3}. \end{aligned}$$

- We set $s_0 = 1$.
- For $\mu = n-1$ we get $B = (2n-1)s_1$ as expected.
- Thus all $s_2, \cdots, s_n$, $X(z)$, are determined by s_1, i.e. by B, alone.
- In fact, a slightly more work shows that the set $a = \{a_i\}$ is also determined by B up to sign. Hence $a \mapsto B_a$ is 2 to 1. **QED**

Theorem (Chai-Lin-W 2012)

- *There is a natural projective compactification* $\bar{X}_n \subset \mathrm{Sym}^n E$ *as a, possibly singular, hyperelliptic curve defined by*

$$C^2 = \ell_n(B, g_2, g_3) = 4Bs_n^2 + 4g_3 s_{n-2} s_n - g_2 s_{n-1} s_n - g_3 s_{n-1}^2,$$

in affine coordinates (B, C)*, where*

$$s_k = s_k(B, g_2, g_3) = r_k B^k + \cdots \in \mathbb{Q}[B, g_2, g_3]$$

is an universal polynomial of homogeneous degree k with $\deg g_2 = 2$*,* $\deg g_3 = 3$*, and* $B = (2n-1)s_1$*.*

- *Thus* $\deg \ell_n = 2n + 1$ *and* $\bar{X}_n$ *has arithmetic genus* $g = n$*.*
- *The curve* $\bar{X}_n$ *is smooth except for a finite number of* τ*, namely the discriminant loci of* $\ell_n(B, g_2, g_3)$*, so that* $\ell_n(B)$ *has multiple roots. In particular* $\bar{X}_n$ *is smooth for rectangular tori.*

(Continued.)

- The $2n+2$ branch points $a \in \bar{X}_n \setminus X_n$ are characterized by $-a = a$.
- $\{-a_i\} \cap \{a_i\} \neq \emptyset \Rightarrow -a = a$.
- Also $0 \in \{a_i\} \Rightarrow a = (0, 0, \cdots, 0)$.
- By setting $(x_i, y_i) = (t_i^3, 2t_i^2)$, the limiting system at $a = 0^n$:

$$\sum_{i=1}^n t_i^{2r+1} = 0, \quad r = 1, \ldots, n-1,$$

  has a unique solution with $t_i \neq 0$ and $t_i \neq -t_j$ in $\mathbb{P}^{n-1}$ up to permutations.

- **Meaning of C (I):** Applying Cramer's rule to the $n-1$ linear equations $\sum_{i=1}^n x_i^k y_i = 0$ in y_i's, there is a constant $C \in \mathbb{C}^\times$ such that

$$y_i = \frac{C}{\prod_{j \neq i}(x_i - x_j)}, \qquad i = 1, \ldots, n.$$

- **Meaning of C (II):** Let w_1, w_2 be two ind. solutions of $w'' = Iw$.

$$C := \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix} = w_1 w_2' - w_2 w_1'$$

  is a (non-zero) constant since $C' = 0$.
- If $X = w_1 w_2$ is known, we may solve w_1, w_2 from C and X:

$$\frac{X'}{X} = \frac{w_1'}{w_1} + \frac{w_2'}{w_2}, \qquad \frac{C}{X} = \frac{w_2'}{w_2} - \frac{w_1'}{w_1},$$

$$\frac{w_1'}{w_1} = \frac{X' - C}{2X}, \qquad \frac{w_2'}{w_2} = \frac{X' + C}{2X}.$$

- In particular

$$w_1 = X^{1/2} \exp\left(-C \int \frac{dz}{2X} \right), \qquad w_2 = X^{1/2} \exp\left(C \int \frac{dz}{2X} \right).$$

- From

$$\left(\frac{X'+C}{2X}\right)' = \left(\frac{w_2'}{w_2}\right)' = \frac{w_2''}{w_2} - \left(\frac{w_2'}{w_2}\right)^2 = I - \frac{(X'+C)^2}{4X^2},$$

  we conclude easily that

$$C^2 = X'^2 - 2X''X + 4IX^2.$$

- The constant terms give the hyperelliptic equation in (B, C).
- In particular, $C = 0$ if and only if $w_a = w_{-a}$, i.e. $a = -a$. These are the branch points of $\bar{X}_n$.
- **Definition:** Denote by $Y_n = \bar{X}_n \backslash \{0^n\}$ the affine hyperelliptic curve defined by

$$C^2 = \ell_n(B, g_2, g_3).$$

- Now we study the last equation on $\bar{X}_n$:

$$0 = -4\pi \sum_{i=1}^{n} \nabla G(a_i) = \sum_{i=1}^{n} Z(a_i). \tag{2}$$

- Consider the rational function on E^n:

$$\mathbf{z}_n(a_1, \ldots, a_n) := \zeta(a_1 + \cdots + a_n) - \sum_{i=1}^{n} \zeta(a_i).$$

(It is periodic in each variable.)

- Let $a_i = t_i\omega_1 + s_i\omega_2$, then

$$\begin{aligned} -4\pi \sum \nabla G(a_i) &= \sum Z(a_i) = \sum(\zeta(a_i) - t_i\eta_1 - s_i\eta_2) \\ &= \zeta(\sum a_i) - (\sum t_i)\eta_1 - (\sum s_i)\eta_2 - \mathbf{z}_n(a) \\ &= Z(\sum a_i) - \mathbf{z}_n(a). \end{aligned}$$

Hence (2) is equivalent to

$$\mathbf{z}_n(a) = Z(\sum a_i). \tag{3}$$

- It is thus crucial to study the branched covering map

$$\sigma : \bar{X}_n \to E, \qquad a \mapsto \sigma(a) := \sum_{i=1}^{n} a_i.$$

Theorem (Lin–W 2013, new pre-modular functions)

(1) *The map σ has degree equals $\frac{1}{2}n(n+1)$.*

(2) *There is a universal (weighted homogeneous) polynomial $W_n(x) \in \mathbb{C}[g_2, g_3, \wp(\sigma), \wp'(\sigma)][x]$ of degree $\frac{1}{2}n(n+1)$ such that*

$$W_n(\mathbf{z}_n) = 0.$$

In fact, $\mathbf{z}_n \in K(\bar{X}_n)$ is a primitive generator for the field extension $K(\bar{X}_n)$ over $K(E)$.

(3) *The function $Z_n(\sigma; \tau) := W_n(Z)$ is pre-modular of weight $\frac{1}{2}n(n+1)$. That is, it is modular wrt. $\Gamma(N)$ if $\sigma \in E_\tau[N]$.*

- **Idea of proof for (1):** Apply *Theorem of the Cube*: For any three morphisms $f, g, h : V_n \longrightarrow E$ and $L \in \operatorname{Pic} E$,

$$(f+g+h)^*L \cong (f+g)^*L \otimes (g+h)^*L \otimes (h+f)^*L$$
$$\otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

- Apply to the case $V_n \subset E^n$ which is the ordered n-tuples so that $V_n/S_n = \bar{X}_n$, and $\deg L = 1$. We prove inductively that the map

$$f_k(a) := a_1 + \cdots + a_k$$

has degree $\frac{1}{2}k(k+1)n!$. It is not hard to check for $k = 1, 2$.

- From k to $k+1$, we let $f = f_{k-1}$, $g(a) = a_k$, and $h(a) = a_{k+1}$.
- Then f_{k+1} has degree $n!$ times

$$\tfrac{1}{2}k(k+1) + 3 + \tfrac{1}{2}k(k+1) - \tfrac{1}{2}(k-1)k - 1 - 1 = \tfrac{1}{2}(k+1)(k+2).$$

- **Idea of proof of (2):** Major tool: *tensor product* of two Lamé equations $w'' = I_1 w$ and $w' = I_2 w$, where $I = n(n+1)\wp(z)$, $I_1 = I + B_a$ and $I_2 = I + B_b$.
- For $\bar{X}_n(\tau)$ smooth, and a general point $\sigma_0 \in E$, we need to show that the $\frac{1}{2}n(n+1)$ points on the fiber of $\bar{X}_n \to E$ above σ_0 has distinct $\mathbf{z}_n$ values. It is enough to show that for $\sigma(a) = \sigma(b) = \sigma_0$, the condition $\sum \zeta(a_i) = \sum \zeta(b_i)$ implies $B_a = B_b$ (and then $a = b$).
- If $w_1'' = I_1 w_1$ and $w_2'' = I_2 w_2$, then the product $q = w_1 w_2$ satisfies

$$q'''' - 2(I_1 + I_2)q'' - 6I'q' + ((B_a - B_b)^2 - 2I'')q = 0.$$

- If $a \neq b$, by addition law we find that $Q = w_a w_{-b} + w_{-a} w_b$ is an *even elliptic function* solution, namely a *polynomial* in $x = \wp(z)$. This leads to strong constraints on the corresponding 4-th order ODE in variable x, and eventually leads to a contradiction for generic choices of σ_0.

Indeed,

$$
\begin{aligned}
&p(x)^2 \ddddot{q} + 3p(x)\dot{p}(x)\dddot{q} \\
&\quad + \left(\tfrac{3}{4}\dot{p}(x)^2 - 2(2(n^2+n-12)x + B_a + B_b)p(x)\right)\ddot{q} \\
&\qquad - \left((2(n^2+n-3)x + B_a + B_b)\dot{p}(x) + 6(n^2+n-2)p(x)\right)\dot{q} \\
&\qquad\quad + \left((B_a - B_b)^2 - n(n+1)\dot{p}(x)\right)q = 0.
\end{aligned}
\tag{4}
$$

As an even elliptic function, Q takes the form

$$
\begin{aligned}
Q(x) &= C\prod_{i=1}^{n}(\wp(z) - \wp(c_i)) =: C\prod_{i=1}^{n}(x - x_i) \\
&= C(x^n - s_1x^{n-1} + s_2x^{n-2} - \cdots + (-1)^n s_n),
\end{aligned}
$$

The x^{n+2} terms agree automatically. The x^{n+1} degree gives

$$
\sum \wp(c_i) = s_1 = \tfrac{1}{2}\frac{B_a + B_b}{2n-1} = \tfrac{1}{2}\left(\sum \wp(a_i) + \sum \wp(b_i)\right).
$$

- Inductively the x^{n+2-i} coefficient in (4) gives recursive relations to solve s_i interns of $B_a + B_b$, $(B_a - B_b)^2$ and g_2, g_3 for $i = 1, \ldots, n$.
- Indeed

$$s_i = s_i(B_a + B_b, (B_a - B_b)^2, g_2, g_3) = C_i(B_a + B_b)^i + \cdots$$

  is homogeneous of degree i if we assign $\deg B_a = \deg B_b = 1$ and $\deg g_2 = 2$, $\deg g_3 = 3$.
- There are two remaining consistency equations $F_1 = 0, F_0 = 0$ coming from the x^1 and x^0 coefficients in (4).
- In fact $(B_a - B_b)^2$ is a factor of both equations and we may write $F_1(B_a, B_b) = (B_a - B_b)^{2d_1} G_1(B_a, B_b)$ and $F_0(B_a, B_b) = (B_a - B_b)^{2d_0} G_0(B_a, B_b)$.
- If $B_a \neq B_b$ (i.e $\sum \wp(a_i) \neq \sum \wp(b_i)$), then

$$G_1(B_a, B_b) = 0, \qquad G_0(B_a, B_b) = 0,$$

  which has only a finite number of solutions (B_a, B_b)'s, i.e. E_τ's.

Example (of compatibility equations for $n = 2$)

For $n = 2$ we have $s_1 = \frac{1}{6}(B_a + B_b)$ and

$$s_2 = \frac{1}{36}(B_a + B_b)^2 + \frac{1}{72}(B_a - B_b)^2 - \frac{1}{4}g_2.$$

The first compatibility equation from x^1 is (after substituting s_1)

$$\frac{1}{6}(B_a - B_b)^2(B_a + B_b) = 0.$$

The second compatibility equation from x^0 is

$$(B_a - B_b)^2(\frac{1}{36}(B_a + B_b)^2 + \frac{1}{72}(B_a - B_b)^2 - \frac{1}{6}g_2) = 0.$$

If $B_a \neq B_b$ then $B_b = -B_a$ and then we can solve B_a, B_b:

$$B_a^2 = 3g_2 \implies \wp(a_1) + \wp(a_2) = \pm\sqrt{g_2/3}.$$

Such $a \in \bar{X}_2$ indeed lies at the branch loci of the Lamé curve.

Example (of new pre-modular forms for $n = 2$)

For $\mathbf{z}_2(a_1, a_2) = \zeta(a_1 + a_2) - \zeta(a_1) - \zeta(a_2)$, on X_2:

$$\mathbf{z}_2^3(a) - 3\wp(a_1 + a_2)\mathbf{z}_2(a) - \wp'(a_1 + a_2) = 0.$$

On E^2 it has one more term $-\frac{1}{2}(\wp'(a_1) + \wp'(a_2))$. Thus,

$$Z_2(\sigma; \tau) = W_2(Z) = Z^3 - 3\wp(\sigma)Z - \wp'(\sigma).$$

Example ($n = 3$)

For $\mathbf{z} = \mathbf{z}_3(a) = \zeta(a_1 + a_2 + a_3) - \zeta(a_1) - \zeta(a_2) - \zeta(a_3)$, on X_3:

$$\mathbf{z}^6 - 15\wp\mathbf{z}^4 - 20\wp'\mathbf{z}^3 + (\tfrac{27}{4}g_2 - 45\wp^2)\mathbf{z}^2 - 12\wp'\wp\mathbf{z} - \tfrac{5}{4}\wp'^2 = 0.$$

Thus, $Z_3(\sigma; \tau) = W_3(Z)$.

- **Key point:** $Z_1 \equiv Z = -4\pi\nabla G$ is the Hecke modular function. The critical point equation ($\Longleftrightarrow$ type II solutions of MFE) is transformed into zero of pre-modular forms.
- For general $n \geq 1$, we have the equivalences:
  - Solution u to MFE for $\rho = 8\pi n$.
  - Periods integral $\int_{L_j} g \in \sqrt{-1}\mathbb{R}$ ($= \omega_j$ coordinates of $\sum a_i$.)
  - Green equation $\sum_{i=1}^{n} \nabla G(a_i) = 0$ on X_n.
  - $\mathbf{z}_n(a) = Z(\sigma(a))$.
  - Non-trivial zero of $Z_n(\sigma;\tau) := W_n(Z)$.
- Need to prove the last one. Notice that the branch point $a \in Y_n \backslash X_n$ ($a \neq -a$) satisfies the Green equation trivially.

- The second technique used in $\rho = 8\pi$ is to use the *method of continuity* to connect to the known case $\rho = 4\pi$ by establishing the non-degeneracy of linearized equations.
- For general ρ, such a non-degeneracy statement is out of reach. However, since solutions u_η always exist for $\rho = 8\pi\eta$, $\eta \notin \mathbb{N}$, it is natural to study the limiting behavior of u_η as $\eta \to n$. If the limit does not blow up, it converges to a solution u for $\rho = 8\pi n$.
- For the blow-up case, we have the connection between the blow-up set and the hyperelliptic geometry of $Y_n \to \mathbb{P}^1$:

Theorem

Suppose that $S = \{p_1, \cdots, p_n\}$ is the blow-up set of a sequence of solutions u_k to with $\rho_k \to 8\pi n$ as $k \to \infty$, then $S \in Y_n$. Moreover,

(1) *If $\rho_k \neq 8\pi n$ then S is a branch point ($a = -a$) of Y_n.*
(2) *If $\rho_k = 8\pi n$ for all k, then S is not a branch point of Y_n.*

- To go deeper, need to know the converse statement: for which $p \in Y_n \backslash X_n$ can we construct a blow-up sequence with blow-up set p? The Morse type of p is fundamental.

Theorem

Suppose that the pair of non half-period critical points $\{\pm p\}$ of G exists, the $\pm p$ are the minimal points of G.

- In fact our proof shows that any solution for $\rho = 8\pi$ must be a minimizer of the non-linear functional

$$J_{8\pi}(u) = \frac{1}{2}\int_E |\nabla u|^2 - 8\pi \log \int_E e^{-8\pi G + u}$$

on $u \in H^1(E) \cap \{u \mid \int_E u = 0\}$.

Corollary

For $\tau \in \Omega_5$, all the three half periods are (non-degenerate) saddle points.

- If u_k is a blow-up sequence with $\rho = \rho_k \to 8\pi$ (as $k \to \infty$), $\rho_k \neq 8\pi$ for large k, then the blow-up point q is a half period.
- Asymptotically

$$\rho_k - 8\pi = (D(q) + o(1))e^{-\lambda_k} \tag{5}$$

where $\lambda_k = \max_{E_\tau} u_k$ and

$$D(q) := \int_{E_\tau} \frac{h(z)e^{8\pi(\tilde{G}(z,q)-\phi(q))} - h(q)}{|z-q|^4} - \int_{E_\tau^c} \frac{h(q)}{|z-q|^4}.$$

Here $h(z) = e^{-8\pi G(z)}$, $\tilde{G}(z,q)$ is the regular part of the Green function, and $\phi(q) = \tilde{G}(q,q)$.

- The sign of $D(q)$ determines the direction where the bubbling may take place, namely $\rho_k < 8\pi$ or $\rho_k > 8\pi$.

Theorem (Lin–W)

For any half period $q \in E_\tau$, $\tau = a + bi$, we have

$$D(q) = -4\pi^2 b e^{-8\pi G(q)} \det D^2 G(q). \qquad (6)$$

- Hence $D(q) > 0$ if q is a saddle point. In particular if $\tau \in \Omega_5$ then $D(q) > 0$ for all half-periods since they are all saddle.
- Since the extra critical point p (reps. $-p$) is a discrete minimal point, the index of ∇G at p (reps. $-p$) is 1. By the Hopf–Poincaré index theorem,

$$-1 = \chi(E_\tau \backslash \{0\}) = 2 + \sum_{i=1}^{3} \operatorname{ind}_{\frac{1}{2}\omega_i} \nabla G.$$

Since $\frac{1}{2}\omega_i$ is non-degenerate, ∇G has index ± 1 at it. Hence the index must be -1 for all $i = 1, 2, 3$. This implies that $\frac{1}{2}\omega_i$ is a saddle point for all i.

- Combining with a recent technique in analyzing uniqueness of blow-up solutions by Lin–Yan, we may classify all solutions to the mean field equation for $\rho \in (0, 8\pi + \epsilon_0)$ for some $\epsilon_0 > 0$:

Theorem (Lin–W)

(i) *If* $\tau \in \Omega_3$ *then the MFE has only one solution for* $\rho < 8\pi$*, no solution for* $\rho = 8\pi$*, and two solutions for* $8\pi < \rho < 8\pi + \epsilon_0$.

(ii) *If* $\tau \in \Omega_5$ *then the MFE has only one solution for* $\rho < 8\pi$*, infinitely many solutions for* $\rho = 8\pi$*, and four solutions for* $8\pi < \rho < 8\pi + \epsilon_0$.

- MFE with $\rho = 12\pi$ has exactly two solutions on E_τ for $\tau \neq e^{\pi i/3}$. Hence when $\tau \in \Omega_5$ the bifurcation diagram is complicate for $\rho \in (8\pi, 12\pi)$. It is a natural question whether MFE has exactly two solutions for $\rho \in (8\pi, 16\pi)$ when $\tau \in \Omega_3$.
- The Theorem also reflects the difficulty in the study the corresponding Lamé equation for the case $\eta \notin \frac{1}{2}\mathbb{N}$.

- The hyperelliptic curve Y_n is parametrized by (B, C) with $C^2 = \ell_n(B)$. In particular, near a branch point p we can use C as the coordinate of Y_n.
- Let $(\partial a_i / \partial C|_{C=0})_{i=1}^n$ be the tangent vector at p and set

$$c_i = 2\frac{\partial a_i}{\partial C}\Big|_{C=0}, \quad s = \sum_{i=1}^n c_i, \quad c_0 = -\sum_{i=1}^n \wp(p_i) c_i.$$

- As in the case $n = 1$, these two invariants are related to a geometric quantity $D(p)$ derived from the blow-up analysis of solutions u_k with $\rho_k \to 8\pi n$. Let $p = (p_1, \cdots, p_n)$ with $\{p_1, \cdots, p_n\}$ being the blow-up set of u_k. Then

$$\rho_k - 8\pi n = (D(p) + o(1))e^{-\lambda_k}, \qquad \lambda_k := \max_E u_k.$$

- The analytic expression of $D(p)$ is rather complicate. However, its geometric meaning is reflected in the following

Theorem

For any branch point $p \in Y_n$, there is a constant $C(p) > 0$ such that

$$D(p) = C(p)|s|^2\left(\left|\frac{c_0}{s} - \eta_1\right|^2 + \frac{2\pi}{b}\mathrm{Re}\left(\frac{c_0}{s} - \eta_1\right)\right).$$

Let $G_n(z_1, \cdots, z_n) := \sum_{i<j} G(z_i - z_j) - n\sum_{i=1}^{n} G(z_i)$. It can be shown that $a = (a_1, \cdots, a_n)$ is a solution to the algebraic/Green system if and only if $z = a$ is a critical point of $G_n(z)$.

Conjecture

For $n \in \mathbb{N}$ and $p = (p_1, \cdots, p_n) \in Y_n \backslash X_n$, there is a $c_p \geq 0$ such that

$$\det D^2 G_n(p) = (-1)^n c_p D(p).$$

Moreover, $c_p > 0$ except for a finite set of tori.

(This has been verified for $n = 1, 2$.)