A Quantum Splitting Principle

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Quantum cohomology

- ► Let *X* be smooth projective variety over C.
- ▶ Basis $T_i \in H = H(X)$, dual $\{T^i\}$, $t = \sum t^i T_i$, $g_{ij} := \langle T_i, T_j \rangle$.
- Genus zero GW formal prepotential $F(t) = \langle \langle \rangle \rangle$:

$$\langle \langle a_1,\ldots,a_m \rangle \rangle = \sum_{\beta \in NE(X)} \sum_{n=0}^{\infty} \frac{q^{\beta}}{n!} \langle a_1,\ldots,a_m,t^{\otimes n} \rangle_{g=0,m+n,\beta}.$$

► 3-pt function $F_{ijk} = \partial^3_{ijk} F = \langle \langle T_i, T_j, T_k \rangle \rangle$, $A^k_{ij} := F_{ijl} g^{lk}$, then $T_i *_t T_j = \sum A^k_{ij}(t) T_k$.

• The Dubrovin connection ∇ on $T_0 \hat{H} \otimes \mathbb{C}[\![q^\bullet]\!] \times \mathbb{A}^1_z$ is flat:

$$abla = d - rac{1}{z} \sum_i dt^i \otimes A_i = d - rac{1}{z} \sum_i dt^i \otimes T_i *_t.$$

Gromov–Witten invariants

Let $\mathcal{M}_{g,n}(X,\beta)$ be the moduli stack of *n*-pointed genus *g* stable maps $f: (C; x_1, \ldots, x_n) \to X$ with $f_*[C] = \beta \in H_2(X)$. We have

$$\operatorname{ev}_j : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X, \quad f \mapsto f(x_j), \quad 1 \le j \le n$$

For $\alpha_j \in H^*(X)$, $\psi_j = c_1(x_j^* \omega_{\mathscr{C}/\overline{\mathscr{M}}_{g,n}(X,\beta)})$, the *descendant invariant* is

$$\Big\langle \prod_{j=1}^n \tau_{k_j}(\alpha_j) \Big\rangle_{g,\beta}^X = \int_{[\overline{\mathscr{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_j \mathrm{ev}_j^*(\alpha_j) \prod_j \psi_j^{k_j}.$$

When $2g + n \ge 3$, there is a stabilization map

$$\operatorname{st}: \overline{\mathscr{M}}_{g,n}(X,\beta) \to \overline{\mathscr{M}}_{g,n}.$$

Now let $\bar{\psi}_j \in H^2(\overline{\mathcal{M}}_{g,n})$ instead. Then the *ancestor invariant* is

$$\left\langle \prod_{j=1}^{n} \bar{\tau}_{k_{j}}(\alpha_{j}) \right\rangle_{g,\beta}^{X} = \int_{[\overline{\mathscr{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{j} \mathrm{ev}_{j}^{*}(\alpha_{j}) \operatorname{st}^{*}(\prod_{j} \bar{\psi}_{j}^{k_{j}}).$$

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Cyclic \mathcal{D}^z -modules

►
$$t = t_0 + t_1 + t_2, t_0 \in H^0, t_1 \in H^2$$
:

$$J(t, z^{-1}) = 1 + \frac{t}{z} + \sum_{\beta, n, i} \frac{q^{\beta}}{n!} T_i \left\langle \frac{T^i}{z(z - \psi_1)}, (t)^n \right\rangle_{\beta}$$

$$= e^{\frac{t}{z}} + \sum_{\beta \neq 0, n, i} \frac{q^{\beta}}{n!} e^{\frac{t_0 + t_1}{z} + (t_1 \cdot \beta)} T_i \left\langle \frac{T^i}{z(z - \psi_1)}, (t_2)^n \right\rangle_{\beta}.$$

► TRR \implies QDE (denote by $z\partial_i = z\partial_{t^i} = z\partial_{T_i}$):

$$z\partial_i z\partial_j J = \sum_k A^k_{ij}(t) \, z\partial_k J.$$

• $QH(X) \equiv \text{cyclic } \mathscr{D}^z \text{-module } \mathscr{D}^z J \text{ with basis (frame)}$

 $z\partial_i J \equiv e^{t/z} T_i \pmod{q^{\bullet}} = T_i + \cdots$

<□ > < 部 > < 注 > < 注 > 注 の Q ペ 5/34 Given an ordinary *P*^{*r*}-flop

$$f: X \dashrightarrow X',$$

the graph $\Gamma_f \subset X \times X'$ induces an isomorphism of motives

$$\mathscr{F} = [\bar{\Gamma}_f]_* : H(X) \xrightarrow{\sim} H(X'),$$

which preserves the Poincaré pairing. We set

$$\mathscr{F}(q^{\beta}) = q^{\mathscr{F}(\beta)}$$

Theorem (Analytic continuation in q^{ℓ})

The correspondence \mathscr{F} induces an isomorphism of big quantum rings $QH(X) \cong QH(X')$ after an analytic continuation over the Novikov variable q^{ℓ} corresponding to the extremal ray. The results also hold for relative invariants and (relative) ancestors.

Step 1 [LLW 2008]

- Degeneration + Reconstruction reduce the proof to the case of *local models*.
- Let (S, F, F') consist of two v.b.'s F and F' of rank r + 1 over a smooth S. The *f*-exc loci Z ⊂ X and Z' ⊂ X' are

$$\bar{\psi}: Z = P_S(F) \to S, \qquad \bar{\psi}': Z' = P_S(F') \to S,$$

and the (projective) local model of f is

$$X = P_Z(N \oplus \mathscr{O}) \xrightarrow{f} X' = P_{Z'}(N' \oplus \mathscr{O}),$$

where $N = N_{Z/X} \cong \mathscr{O}_Z(-1) \otimes \overline{\psi}^* F'$ and similarly for N'.

- The flop *f* is the blowup of *X* along *Z* followed by contracting the exc-divisor $E = Z \times_S Z'$ along the $\overline{\psi}$ -ruling.
- ► The local model of *f* is a functor over the triples (*S*, *F*, *F*')'s.

Step 2 [LLW 2011]

▶ For $F = \bigoplus_{i=0}^{r} L_i$, $F' = \bigoplus_{i=0}^{r} L'_i$ being split bundles, based on [Brown 2009], a quantum Leray–Hirsch theorem is proved:

$$QH(X) \cong_{\mathscr{D}^z} p^*QH(S)[\hat{h},\hat{\xi}]/(\hat{f}_F,\hat{f}_{N\oplus\mathscr{O}}).$$

• Here
$$\hat{h} = z\partial_h$$
, $\hat{\xi} = z\partial_{\xi}$, and

$$\hat{f}_F = \Box_{\ell} = \prod z \,\partial_{h+L_i} - q^{\ell} e^{t^h} \prod z \,\partial_{\xi-h+L'_i},$$
$$\hat{f}_{N \oplus \mathscr{O}} = \Box_{\gamma} = z \,\partial_{\xi} \prod z \,\partial_{\xi-h+L'_i} - q^{\gamma} e^{t^{\xi}},$$

are the Picard–Fuchs operators which are the "quantized version" of the Chern polynomials.

► The pullback p*QH(S) is an admissible lifting of the Dubrovin connection on H(S) to H(X): Let $D = t^h h + t^{\xi} \xi$ be the relative divisor class, $\overline{t} \in H(S)$, then

$$z\partial_i z\partial_j = \sum_{\bar{\beta}\in NE(S),k} q^{\beta} e^{D.\bar{\beta}^*} \left[A_S\right]_{ij,\bar{\beta}}^k(\bar{t}) z\partial_k \mathbf{D}_{\beta}(z)$$

for some *admissible lifting* $\beta \in NE(X)$ and differential operator

$$\mathbf{D}_{\beta}(z) := \prod_{m=0}^{-\xi,\beta-1} (z\partial_{\xi} - mz) \times \prod_{i=0}^{r} \left(\prod_{m=0}^{-(h+L_i),\beta-1} (z\partial_{h+L_i} - mz) \prod_{m=0}^{-(\xi-h+L_i'),\beta-1} (z\partial_{\xi-h+L_i'} - mz) \right)$$

Here β is admissible if $-(h + L_i).\beta \ge 0$, $-(\xi - h + L'_i).\beta \ge 0$ and $-\xi.\beta \ge 0$. It exists, but might not be unique. Nevertheless, $\mathbf{D}_{\beta}(z)$ is well-defined modulo the Picard–Fuchs ideal $\langle \Box_{\ell}, \Box_{\gamma} \rangle$. Now we may compute the first order system

$$z\partial_{t^a}(\hat{T}_i) = (\hat{T}_i)C_a(z,q^{\bullet}), \qquad t^a = t^h, t^{\xi}, \overline{t}^i.$$

under the naive frame $\hat{T}_i = z \partial_{\bar{t}^i} (z \partial_{t^{j_i}})^j (z \partial_{t^{\xi_i}})^{k's}$.

- This is "equivalent" to $\mathcal{D}^z J^X$ as \mathcal{D}^z -modules.
- ► The analytic continuation of *D*^z-modules in *q*^ℓ follows easily from the above presentation and

$$\mathscr{F}: \langle \Box_{\ell}, \Box_{\gamma} \rangle \cong \langle \Box_{\ell'}, \Box_{\gamma'} \rangle.$$

- ► To get QH(X) from the D^z-module, we need BF/GMT: Birkhoff factorization/generalized mirror transform.
- A technical induction was performed so that this procedure is compatible with analytic continuations.

Example

Let $f : X \to X'$ be a P^1 -flop, $(S, F, F') = (P^1, \mathcal{O} \oplus \mathcal{O}, \mathcal{O} \oplus \mathcal{O}(1))$. Write $H(S) = \mathbb{C}[p]/(p^2)$ with Chern polynomials

$$f_F(h) = h^2$$
, $f_{N \oplus \mathscr{O}}(\xi) = \xi(\xi - h)(\xi - h + p)$.

Then $H = H(X) = H(S)[h, \xi]/(f_F, f_{N\oplus \mathscr{O}})$ has dimension N = 12 with basis $\{T_i \mid 0 \le i \le 11\}$ being

1, h,
$$\xi$$
, p, h ξ , hp, ξ^2 , ξ p, h ξ^2 , h ξ p, ξ^2 p, h ξ^2 p.

Denote by $q_1 = q^{\ell}e^{t^1}$, $q_2 = q^{\gamma}e^{t^2}$, $\bar{q} = q^{b}e^{t^3}$, where $b = [S] \cong [P^1]$. The Picard-Fuchs operators are

$$\Box_{\ell} = (z\partial_{h})^{2} - q_{1}z\partial_{\xi-h}z\partial_{\xi-h+p},$$
$$\Box_{\gamma} = z\partial_{\xi}z\partial_{\xi-h}z\partial_{\xi-h+p} - q_{2}.$$

They lead to a Grobner basis:

$$(z\partial_{h})^{2} = \mathbf{f}(q_{1}) \left((z\partial_{\xi})^{2} - z\partial_{p} z\partial_{h} + z\partial_{p} z\partial_{\xi} - 2z\partial_{h} z\partial_{\xi} \right),$$

$$(z\partial_{\xi})^{3} = q_{2}(1-q_{1}) - z\partial_{p}(z\partial_{\xi})^{2} + 2z\partial_{h}(z\partial_{\xi})^{2} + z\partial_{p} z\partial_{h} z\partial_{\xi}.$$

Here $f(q) := q/(1 - (-1)^{r+1}q)$ which satisfies

$$\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r.$$

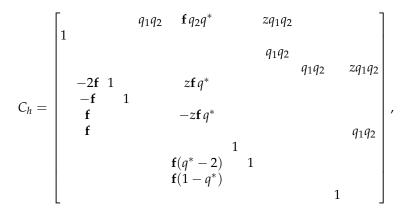
 $H(S) = \mathbb{C}\mathbf{1} \oplus \mathbb{C}p$ has only small parameter \bar{q} with QDE

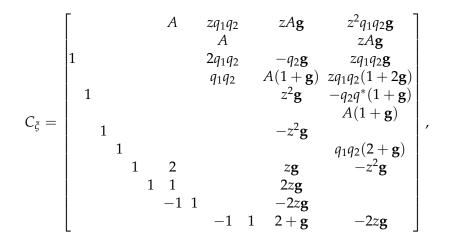
$$z\partial_p(z\partial_1, z\partial_p) = (z\partial_1, z\partial_p) \begin{pmatrix} 0 & \bar{q} \\ 1 & 0 \end{pmatrix}$$

We have admissible lifting $b^I = b - \gamma$ and $\mathbf{D}_b = z \partial_{\xi} z \partial_{\xi - h}$, hence the lifted QDE:

$$(z\partial_p)^2 = \bar{q}q_2^{-1} z\partial_{\xi} z\partial_{\xi-h}.$$

We calculate C_a in $z\partial_a \hat{T}_j = \sum_k C_{aj}^k(z)\hat{T}_k$. Let $q^* = \bar{q}q_2^{-1}$ be the chosen admissible lift. Set $\mathbf{g} = \mathbf{f}(q^*), A = q_2 - q_1q_2, S = q_2 + q_1q_2$. Then





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and $C_p =$

A gauge transform is needed to remove all appearances of *z*. In this example GMT is not needed since the first column vectors in C_a 's are correct: $\hat{T}_i * \hat{\mathbf{1}} = \hat{T}_i$.

Step 3: splitting principle [LLQW 2014]

Proposition

Given a \mathbb{C}^k -bundle $F \to S$, there exists a sequence of blow-ups on smooth centers $\phi : \tilde{S} \to S$ such that there is a filtration of subbundles

$$0=F_0\subset F_1\subset\ldots\subset F_k=\phi^*F$$

with $\operatorname{rk} F_{i+1}/F_i = 1$ for all *i*; ϕ^*F can be deformed to a split bundle.

Proof.

Consider the *complete flag bundle* over *S* and a *rational section s*:

$$\mathcal{F}_S(F) \xrightarrow[s]{p} S.$$

Let $\phi : \tilde{S} \to S$ resolves *s*. Then ϕ^*F admits a complete flag, and there is a deformation of sending all extension classes to 0.

In the classical setting

 $p^*: H(S) \hookrightarrow H(\mathcal{F}_S(F)), \qquad \phi^*: H(S) \hookrightarrow H(\tilde{S})$

are both ring monomorphisms.

- They lead to the *classical splitting principle*.
- Such functorialties fail for *QH*.
- Instead, we develop a quantum splitting principle to study

$$QH(S) \dashrightarrow QH(\mathcal{F}_S(F)), \qquad QH(S) \dashrightarrow QH(\tilde{S}).$$

► In particular, *F*-invariance (analytic continuations)

$$\mathscr{F}: QH(X_{(S,F,F')}) \cong QH(X'_{(S,F,F')})$$

with $\mathscr{F}q^{\ell} = (q^{\ell'})^{-1}$ is reduced to the split case.

Starting with $(S_0, F_0, F'_0) = (S, F, F')$, we construct $(S_i, F_i, F'_i)_{i \ge 0}$:

$$\phi_i: S_{i+1} = \operatorname{Bl}_{T_i} S_i \to S_i$$

for some smooth $T_i \subset S_i$, $F_{i+1} = \phi_i^* F_i$ and $F'_{i+1} = \phi_i^* F'_i$.

- ℱ-invariance for (S_i, F_i, F'_i) can be reduced to the ℱ-invariance for the triple in the next stage (S_{i+1}, F_{i+1}, F'_{i+1}).
- ► The problem is solved for S_{i+1} = Š since GW theory is invariant under smooth deformations.

We consider the deformation to the normal cone for $T_i \hookrightarrow S_i$:

$$\Phi_i : \mathbf{S} = \operatorname{Bl}_{T_i \times \{0\}}(S_i \times \mathbb{A}^1) \to \mathbb{A}^1,$$
$$\mathbf{S}_t = S_i \sim S_{i+1} \cup_{E_i} P_i = \mathbf{S}_0,$$
$$E_i = \operatorname{Exc} \phi_i = P_{T_i}(N_{T_i/S_i}), \text{ and } P_i = \operatorname{Exc} \Phi_i = P_{T_i}(N_{T_i/S_i} \oplus \mathscr{O}).$$

For simplicity, we write

$$X_{S_i} \equiv X_{(S_i, F_i, F_i')}$$

etc. when the bundles are from pullbacks (restrictions).The degeneration formula in GW theory says that

$$\langle \alpha \rangle^{X_{S_i}} = \sum_{\vec{\mu}} \langle \alpha_1 \mid \vec{\mu} \rangle^{\bullet(X_{S_{i+1}}, X_{E_i})} \langle \alpha_2 \mid \vec{\mu}^{\vee} \rangle^{\bullet(X_{P_i}, X_{E_i})}$$

where $\vec{\mu} = \{(\mu_i, e_i)\}$ is a $H(X_{E_i})$ -weighted partition.

Thus, for both factors, we need to control

relative invariants for a smooth divisor pair (X_S, X_D)

by the *absolute invariants* of X_S and X_D .

A trivial degeneration (to the normal cone)

$$S \sim S \cup_D P, \qquad P = P_D(N \oplus \mathscr{O}) \xrightarrow{\pi} D$$

leads to

$$\langle \alpha \rangle^{X_{S}} = \sum_{\vec{\mu}} \langle \alpha_{1} \mid \vec{\mu} \rangle^{\bullet(X_{S}, X_{D})} \langle \alpha_{2} \mid \vec{\mu}^{\vee} \rangle^{\bullet(X_{P}, X_{D})}$$

- The problem becomes "inversion of this linear system", with coefficients being relative invariants of (X_P, X_D).
- Here $X_P \to X_D$ is a split P^1 -bundle arising from $\pi : P \to D$.
- Since $D = P_T(N_{T/S}) \rightarrow T$ has

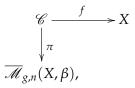
$\dim T < \dim S.$

 \implies the absolute invariants for X_P are handled inductively.

- ► To handle (X_P, X_D), *fiberwise localization* was used in [Maulik–Pandharipande 2006].
- ► Among other technical issues, localizations create *descendants* which breaks *ℱ*-invariance.
- ► We replaced descendants by *descendants of special type*, which solved the simple P^r-flop case in [LLW 2006].
- And then by *ancestors* in [Iwao–LLW 2012], we extended \mathscr{F} -invariance to all $g \ge 0$ under simple P^r -flops.
- ▶ Now, to treat general $P = P_D(N \oplus O)$, localizations are replaced by *more complex degeneration argument* and
- the strong virtual pushforward property, which extends earlier works of [H.-H. Lai 2008, Manolache 2012].

Review of relative obstruction theory

The universal curve $\mathscr{C} = \overline{\mathscr{M}}_{g,n+1}(X,\beta)$ with $f = ev_{n+1} : \mathscr{C} \to X$:



leads to a perfect obstruction theory and virtual cycle

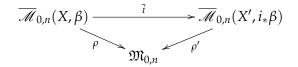
$$E^{\bullet} := (R\pi_* f^* T_X)^{\vee} \to \mathbb{L}_{\overline{\mathscr{M}}}, \text{ and } [\overline{\mathscr{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$$

[Li–Tian 1998, Behrend–Fantachi 1997]. Also a relative theory for $i : X \hookrightarrow X'$ OR with $i_* : A_1(X) \hookrightarrow A_1(X')$ [Manolache 2012]:

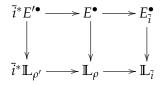
$$\overline{i}: \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X',i_*\beta),$$
$$E^{\bullet}_{\overline{i}} := (R\pi_*f^*\mathbb{L}_i^{\vee})^{\vee} \to \mathbb{L}_{\overline{i}}.$$

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It is perfect if g = 0 and X' is convex (e.g. homogeneous). Then there is a commutative diagram



through the Artin stack $\mathfrak{M}_{0,n}$ of prestable curves. We have compatible obstruction theories



and $[\overline{\mathscr{M}}_{0,n}(X,\beta)]^{\mathrm{vir}} = \overline{i}! [\overline{\mathscr{M}}_{0,n}(X',i_*\beta)]^{\mathrm{vir}}.$

Strong Virtual Pushforward

• Consider the split *P*¹-bundle

 $\pi: Y = P_X(L \oplus \mathscr{O}) \to X,$

which has two sections $i_0 : Y_0 \hookrightarrow Y$ and $i_\infty : Y_\infty \hookrightarrow Y$.

- ► Relative/Log GW invariants on (Y, Y₀) and (Y, Y_∞) are called *type I*; those on (Y, Y₀ ⊔ Y_∞) are called *type II*. They are equivalent for g = 0 [Abramovich et. al 2014].
- Let (Y, Y₀ ⊔ Y_∞), (Y, Y₀) and (Y, Y_∞) denote the log schemes, which are log smooth and integral. And

$$\overline{\mathscr{M}}_{0,n}(Y;\mu,\nu):=\overline{\mathscr{M}}_{0,n}((Y,Y_0\sqcup Y_\infty),\beta;\mu,\nu)\quad\text{etc.}$$

be the log stack of stable log maps with curve class β .

µ, ν are partitions of d₀ = ∫_β Y₀ and d_∞ = ∫_β Y_∞, which specify the contact orders of marked points in Y₀ and Y_∞.

When $\theta := \pi_* \beta \neq 0$ or $n \ge 3$, we have induced maps:

$$p: \overline{\mathcal{M}}_{0,n}(Y;\mu,\nu) \to \overline{\mathcal{M}}_{0,n}(X,\theta), q: \overline{\mathcal{M}}_{0,n}(Y;\nu) \to \overline{\mathcal{M}}_{0,n}(X,\theta).$$

Lemma (Virtual dimension count)

1. dim
$$[\overline{\mathcal{M}}_{g,n}(Y;\mu,\nu)]^{\text{vir}} = \dim [\overline{\mathcal{M}}_{g,n}(X,\theta)]^{\text{vir}} + 1 - g.$$

2. dim $[\overline{\mathcal{M}}_{g,n}(Y;\nu)]^{\text{vir}} = \dim [\overline{\mathcal{M}}_{g,n}(X,\theta)]^{\text{vir}} + 1 - g + \int_{\beta} Y_0.$

Proof. For log moduli stack we need to impose conditions by the contact orders. In (1) it is $d_0 + d_\infty$ and in (2) it is d_∞ . Now

$$c_1(Y).\beta = (\pi^* c_1(X) + Y_0 + Y_\infty).\beta = c_1(X).\theta + d_0 + d_\infty.$$

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Also $(\dim Y - \dim X)(1-g) = 1-g$.

Proposition (Strong virtual pushforward for g = 0)

1. In
$$A_*(\overline{\mathcal{M}}_{0,n}(X,\theta))$$
, there exists $N(\mu,\nu) \in \mathbb{Q}$ such that

$$p_*[\overline{\mathscr{M}}_{0,n}(Y;\mu,\nu)]^{\operatorname{vir}} = 0,$$

$$p_*([\overline{\mathscr{M}}_{0,n}(Y;\mu,\nu)]^{\operatorname{vir}} \cap \operatorname{ev}_1^*[Y_0]) = N(\mu,\nu)[\overline{\mathscr{M}}_{0,n}(X,\theta)]^{\operatorname{vir}}.$$

2. Assume
$$\int_{\beta} Y_0 \ge 0$$
, then $q_*[\overline{\mathcal{M}}_{0,n}(Y;\nu)]^{\text{vir}} = 0$.

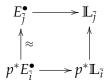
Proof. Choose $M \in \text{Pic } X$ such that M and $L \otimes M$ are both very ample. Then we have a cartesian diagram of embeddings

$$\begin{array}{ccc} Y & \stackrel{j}{\longrightarrow} P(\mathscr{O}(-1,1) \oplus \mathscr{O}) \\ \pi & & & \\ \pi & & & \\ \chi & \stackrel{i}{\longrightarrow} P^{|M|} \times P^{|L \otimes M|}, \end{array}$$

with $L = i^* \mathcal{O}(-1, 1)$. The proposition holds for $\tilde{\pi}$.

It induces a cartesian diagram between log stacks

As \overline{i} is strict, the underlying stack-diagram is also cartesian. The *relative perfect obstruction theories* $E_{\overline{i}}^{\bullet} \to \mathbb{L}_{\overline{i}}$ and $E_{\overline{i}}^{\bullet} \to \mathbb{L}_{\overline{j}}$ fit in



since $p^* \mathbb{L}_i \cong \mathbb{L}_j$ (cf. Manolache). Now $\overline{i}^!$ and $\overline{j}^!$ pullback virtual cycles. The results for (π, p) follow from that for $(\tilde{\pi}, \tilde{p})$.

Back to (X_P, X_D) , i.e. type I invariants

▶ Recall π : $P = P_D(N \oplus \mathscr{O}) \to D$, with $P_0, P_\infty \cong D$, induces

$$\pi: X_P = P_{X_D}(L \oplus \mathscr{O}) \to X_D,$$

with $L = (X_D \to D)^* N$ and sections $X_{P_0}, X_{P_{\infty}} \cong X_D$.

- ► A non-vanishing theorem modelled on (P¹, {0}) × X_{pt} is proved to show the invertibility of the linear system.
- Moreover, under the trivial degenerations

$$P \sim P \cup_{P_{\infty}} P$$
, $(P, P_0) \sim (P, P_0) \cup_{P_{\infty}} P$,

the "strong virtual pushforward" and "TRR for ancestors" \implies type I invariants are determined by absolute, type II, and rubber invariants modulo lower degree ones.

► For $X_P \to Z_P \to P$ with relative hyperplane classes ξ, h , a class $\beta \in NE(X_P)$ has $(\beta_P, d) \in NE(P) \times \mathbb{Z}$ with $d = \int_{\beta} \xi$. The genus 0 generating series

$$\langle \alpha \rangle_{(\beta_P,d)}^{X_P} := \sum_{\beta \in (\beta_P,d)} \langle \alpha \rangle_{\beta}^{X_P} q^{\beta}$$

is a sum over the extremal ray. Similarly for type I, II, etc.
Then for *ω* being pullback insertions from *X*_D, we have

$$\left\langle \vec{v} \mid \omega \cdot \prod_{i=1}^{l} i_{\infty*}(\alpha_i) \right\rangle_{(\beta_P, d)}^{(X_P, X_{P_{\infty}})} = \sum_{I, \eta = (\Gamma_1, \Gamma_2)} C_{\eta} \left\langle \vec{v} \mid \omega_1 \cdot \prod_{i=1}^{l} i_{\infty*}(\alpha_i) \mid \mu, e^I \right\rangle_{\Gamma_1}^{\bullet} \cdot \left\langle \mu, e_I \mid \omega_2 \right\rangle_{\Gamma_2}^{\bullet}$$

spanned by type II and type I series with pullback insertions.

• Moreover, if $\int_{\beta_P} P_0 \ge 0$ then $\langle \omega | \vec{\nu} \rangle_{\beta_S,d}^{(X_P,X_{P_\infty})} = 0$.

Proposition (Type I reduction)

Assume $\int_{\beta_P} P_0 < 0$. An ordering is introduced on $\{\vec{v}\}$ such that 1. If $\vec{v} = \{(v_j, B_j)\} \neq \emptyset$ then there exists $C(\vec{v}) > 0$, $k(\vec{v}) \in \mathbb{Z}_{\geq 0}$,

$$C(\vec{\nu})\langle \omega \mid \vec{\nu} \rangle_{(\beta_{P},d)}^{(X_{P},X_{P_{\infty}})} - \left\langle \omega \cdot [X_{P_{\infty}}]^{k(\vec{\nu})} \cdot \prod_{j} \bar{\tau}_{\nu_{j}-1}(i_{\infty*}(B_{j})) \right\rangle_{(\beta_{P},d)}^{X_{P}}$$

is generated by "relative and rubber series" on X_P of class at most (β_P, d) , and those of $(X_P, X_{P_{\infty}})$ involving class (β_P, d) whose orders are lower than $\langle \omega | \vec{v} \rangle_{(\beta_P, d)}$.

2. If $\vec{v} = \emptyset$ then

$$\langle \omega \mid \vec{v} \rangle_{(\beta_P,d)}^{(X_P,X_{P_{\infty}})} - \langle \omega \rangle_{(\beta_P,d)}^{X_P}$$

is generated by series of relative invariants on X_P with curve classes lower than (β_P, d) .

Theorem (Type II invariance)

 \mathscr{F} -invariance for X_D implies \mathscr{F} -invariance for $(X_P, X_{P_0} \sqcup X_{P_{\infty}})$.

For fiber class inv., i.e. β ∈ NE(X_P/D), they are reduced to the cup product on a birational D' → D and the case

$$(P^1, \{0, \infty\}) \times X_{\mathsf{pt}}.$$

Thus we consider non-fiber class type II-inv.

- Let $k \ge 0$ be the number of non-pullback insertions in $\pi : X_P \to X_D$. If $k \le 1$, the strong pushforward (1) applies.
- If $k \ge 2$, since

$$[X_{P_0}] - [X_{P_\infty}] = \pi^* c_1(N_{X_D/X_P}),$$

modulo type II-inv with k - 1 non-pullback insertions, we may assume one is $i_{0*}(\alpha)$ and the others are $i_{\infty*}(\alpha_i)$.

The family $W = \operatorname{Bl}_{X_{P_{\infty}} \times \{0\}} X_P \times \mathbb{A}^1 \to X_P \times \mathbb{A}^1$ gives

$$\left\langle \begin{array}{l} \vec{\mu} \mid \omega \cdot i_{0*}(\alpha) \prod_{i=1}^{l-1} i_{\infty*}(\alpha_i) \mid \vec{\nu} \end{array} \right\rangle_{(\beta_P, d)}^{(X_P, X_{P_0}, X_{P_\infty})} = \\ \sum_{I, \eta} C_{\eta} \left\langle \begin{array}{l} \vec{\mu} \mid \omega_1 \cdot i_{0*}(\alpha) \mid \lambda, e^I \end{array} \right\rangle_{\Gamma_1}^{\bullet} \left\langle \lambda, e_I \mid \omega_2 \cdot \prod_{i=1}^{l-1} i_{\infty*}(\alpha_i) \mid \vec{\nu} \end{array} \right\rangle_{\Gamma_2}^{\bullet},$$

where $\eta = (\Gamma_1, \Gamma_2)$ is the splitting type.

- Here ω , ω_1 , ω_2 are pullbacks insertions from X_D .
- ► The RHS is determined by type II generating functions with at most *k* − 1 non-pullback insertions.
- ► This relation is compatible with *F*-invariance, and the theorem follows by induction on k ∈ N.

We omit the discussion on rubber calculus. QED

Problem: non-split quantum Leray-Hirsch?

For
$$P = P_X(V) = P_X(\bigoplus_{i=1}^r L_i)$$
, the QLH says that
 $PF^{P/X} + \nabla^X \Longrightarrow \nabla^P$.

For the primitive class $\ell \in NE(P/X)$, the Picard–Fuchs is

$$\Box_{\ell} = \prod_{i=1}^{r} z \partial_{h+L_i} - q^{\ell} e^{t^h}.$$

When *V* is non-split, we may still define $\hat{a} = z\partial_a$ and

$$\widehat{\Box}_{\ell} = \widehat{f}_{V} = \widehat{h}^{r+1} + \widehat{c_{1}(V)}\widehat{h}^{r} + \cdots \widehat{c_{r}(V)} - q^{\ell}e^{t^{h}}$$

Under QDE it is essentially equivalent to \Box_{ℓ} .

For $\bar{\beta} \in NE(X)$, a lift $\beta \in NE(P)$ is *admissible* if $-(h + L_i).\beta \ge 0$ for all *i*. A minimal effective lift $\bar{\beta}^*$ is admissible. In this case

$$\mathbf{D}_{\bar{\beta}^*}(z) := \prod_{i=1}^r \prod_{m=0}^{-(h+L_i),\bar{\beta}^*-1} (z\partial_{h+L_i} - mz).$$

Then the lift of QDE form H(X) to H(P) is

$$z\partial_i z\partial_j = \sum_{k,ar{eta}} q^{ar{eta}^*} e^{D.ar{eta}^*} ar{A}^k_{ij,ar{eta}}(ar{t}) \, \mathbf{D}_{ar{eta}^*}(z) \, z\partial_k.$$

However, $\mathbf{D}_{\bar{\beta}^*}(z)$ depends on $\bar{\beta}^* L_i$ in an essential way and it is unclear if it is equivalent to another expression which does not depend explicitly on the splitting factors L_i 's.

For *V* non-split, our splitting principle do lead to effective determinations of $QH(P_X(V))$ by reducing it to the split case. \Box