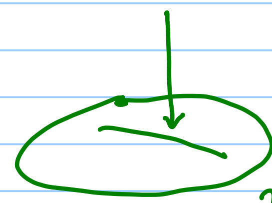
Quantum Invariance under conifold transitions

(8/4 2014) Chin-Lung Wang (joint w H.-W. Lin, Y.-P. Lee)

Given $\begin{array}{c} Y \\ \downarrow \psi \\ \bar{X} \end{array}$ X, Y SU(3) CY 3 folds

$X \rightsquigarrow \bar{X} \ni p_1, \cdots, p_k$ ODP in proj. cat.

$\mathcal{M}_Y$



$\mathcal{M}_{\bar{X}}$

codim μ boundary

Local structure

vanishing cycles / extremal rays $C_i \cong \mathbb{P}^1 \cong S^2 \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)^2$

$\downarrow$

$T^* S^3 \hookleftarrow S^3 \cong S_i \rightsquigarrow p_i$

Topological constraint

$\mu := h^{2,1}(X) - h^{2,1}(Y) = \frac{1}{2}\left(h^3(X) - h^3(Y)\right)$

$\rho := h^{1,1}(Y) - h^{1,1}(X) = h^2(Y) - h^2(X) = \rho(Y/\bar{X})$

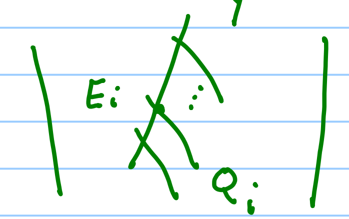
$$\chi(X) - k\,\chi(S^3) = \chi(Y) - k\,\chi(S^2) \Rightarrow \underline{\mu + \rho = k}$$

Relation matrix $A \in M_{k\times\mu}(\mathbb{Z})$, $B \in M_{k\times p}(\mathbb{Z})$ relations for $c_1, \dots, c_k$ / $s_1, \dots, s_k$

Mixed Hodge theory, asso. to two semi-stable reductions

$\tilde{\mathcal{Y}} = Bl_{\cup p_i} \overline{\mathcal{X}}^{(2)}$

$\tilde{\mathcal{Y}}$

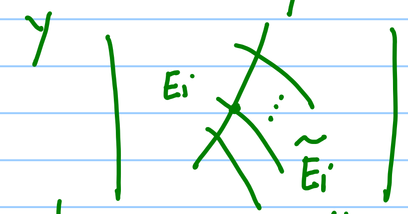


X

$\mathcal{X} \xrightarrow{\pi} \Delta$

Y

$\tilde{\mathcal{Y}}$



$\mathcal{Y} \to \Delta$

Q_i quadratic 3 fold $\supset E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$

$\tilde{E}_i \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O})$

$H^{2,2}_\infty H^3(X)$

$H^{3,0}_\infty H^3(X) \simeq \mathbb{C}$ $\quad H^{2,1}_\infty H^3(X) \quad H^{1,2}_\infty H^3(X) \quad H^{0,3}_\infty H^3(X) \simeq \mathbb{C}$

$H^{2,1}_\infty H^3(X) \xrightarrow{N} H^{1,1}_\infty H^3(X)$

$H^{1,1}_\infty H^3(X) \simeq V :=$ Vanishing cycle space in $H_3(X)$

Thm (Basic Exact sequence, Hodge realization of $\mu + p = k$ over $\mathbb{Z}$)

$$0 \to H^2(Y)/H^2(X) \xrightarrow{B} \bigoplus_{i=1}^{k} H^2(E_i)/H^2(Q_i) \xrightarrow{A^t} V \to 0$$

$B = \ker A^t$

$A = \ker B^t$

Quantum (D modules) Aspects :

A model = Dubrovin connection on Kähler moduli (GW theory)

B model = Gauss-Manin connection on complex moduli (KS theory)

$Y \searrow X$ (or $X \nearrow Y$) $\Rightarrow$ $A(X) < A(Y)$ & $B(X) > B(Y)$

Goal: Determine $(A(X), B(X))$ from $(A(Y), B(Y))$ and vice versa.

$Y \dashrightarrow X$ $A(X)$ det. by $A(Y)$ thr

$$\langle\ \rangle^X_\beta = \sum_{\gamma \mapsto \beta} \langle\ \rangle^Y_\gamma$$

$B(X)$ det. by $A(Y)$ & $B(Y)$ thr 4 steps :   GM on $\begin{array}{c} R^3\pi_*\mathbb{C} \\ \downarrow \\ \mathcal{M}_{\bar{X}} \end{array}$?

1. Setup of coordinates (Friedman)

$A = [A^1, \ldots, A^\mu] = (a_{ij})$. $\forall t \in \mathbb{C}^k$, $A_r := \sum_{j=1}^{\mu} t_j A^j$ (a relation vector)

gives a partial smoothing of all p_i with $A_{ri} \neq 0$

$w_i := A_{ri} = t_1 a_{i1} + \cdots + t_\mu a_{i\mu}$ defines a hyperplane $D^i \subset \mathbb{C}^\mu$, $1 \le i \le k$

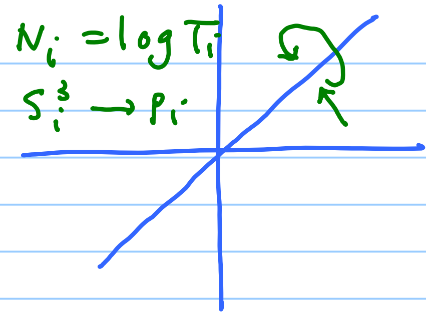
Locally $\mathcal{M}_{\bar{X}} \simeq \Delta^\mu \times \mathcal{M}_Y \ni (t, s)$

2. Transversal Directions

e.g. Δ^μ: D^3 D^2

$N_i = \log T_i$

$S_i^3 \to p_i$



Proposition (Picard-Lefschetz)

May choose $\Omega(t,s)$ and symp. basis $\alpha_0, \underset{\Gamma_1}{\overset{\shortparallel}{\alpha_1}}, \dots, \underset{\Gamma_\mu}{\overset{\shortparallel}{\alpha_\mu}}$, (vanishing cycles) $\alpha_{\mu+1}, \dots, \alpha_{h^{21}}, \beta_0, \dots, \beta_{h^{21}}$ st.

$$\Omega = a_0(s) + \sum_{j=1}^{\mu} \Gamma_j^* t_j + \text{h.o.t.} - \sum_{i=1}^{k} \frac{w_i \log w_i}{2\pi\sqrt{-1}} PD([S_i])$$

$$\underset{3}{\overset{\shortparallel}{k}} = \underset{2}{\overset{\shortparallel}{\mu}} + \underset{1}{\overset{\shortparallel}{\rho}}$$

Corollary (Bryant-Griffiths cubic / Yukawa coupling)

$u_p(t,s) := \int_{\beta_p} \Omega$. for $1 \le p, m, n \le \mu$

$$u_{pmn} = \text{h.o.t.} + \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k} \frac{a_{ip}\, a_{im}\, a_{in}}{w_i}$$

der. by data on Y

otherwise they are non-singular.

3. Boundary Values

∇^{GM} on $H^3(X)$ has conn. matrix 1-form w/ variables (t,s)

$$\overbrace{\begin{pmatrix} \mathcal{A} & \mathcal{C} \\ 0 & \mathcal{B} \end{pmatrix}}^{H^3(X)} \begin{matrix} \} \ 2\mu \text{ dim'l} \ \ V \oplus V^* \\ \} \ H^3(Y) \end{matrix}$$

$\mathcal{A}(t,0)$ given by 2.

$\mathcal{B}(0,s) = B(Y)$ given

$\mathcal{C} = ?$

Gauss-Manin for smooth morphism over non-reduced base

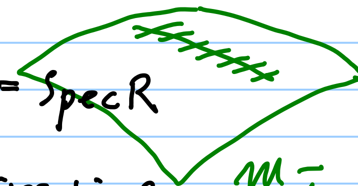
Study Kodaira-Spencer theory on Y "twisted by extr. rays"

$U := Y \setminus Z \simeq \bar{X} \setminus P$ $\quad Z := \bigcup_{i=1}^{k} C_i$, $\quad P := \bigcup_{i=1}^{k} p_i$

$\mathcal{M}_Y \subset S \subset \mathcal{M}_{\bar{X}}$

$\Rightarrow \quad H^1(Y, T_Y) \hookrightarrow H^1(U, T_U) \simeq H^1(\bar{X}, \Theta_{\bar{X}})$

locally $\quad \mathcal{M}_Y = \operatorname{Spec} R/(t_1, \dots, t_\mu) \hookrightarrow S := \operatorname{Spec} R/(t_1, \dots, t_\mu)^2 \hookrightarrow \mathcal{M}_{\bar{X}} = \operatorname{Spec} R$


$\mathcal{M}_{\bar{X}}$

$T_S = H^1(U, T_U)$ integrable in $H^1(T_Y)$, obstructed in other directions

Still get $\pi : \mathcal{U} \to S$ quasi-projective smooth morphism

Katz-Oda (1970's) : ∇^{GM} on $H^3(U, \mathbb{C}) \simeq H^3(X, \mathbb{C})$ is defined over S

$\Rightarrow$ get the off diagonal matrix $\mathcal{C}$ at $t = 0$.

4. Solving $\nabla^2 = 0$ with $\nabla = d_t + d_s + \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ 0 & \mathcal{B} \end{pmatrix}$

$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$

$= \sum \mathcal{B}_1^j \, dt_j + \sum \mathcal{B}_2^\nu \, ds_\nu$

Rigidity of ∇^{GM} on $\mathcal{M}_Y$ (Viehweg-Zuo)

$\Rightarrow \quad \mathcal{B}_2(t, s) = g^{-1}(t) \, \mathcal{B}(s) \, g(t)$, $\quad g(0) = id$ ※

$X \dashrightarrow Y$: $B(Y) \subset B(X)$ as the monodromy invariant part.

$A(Y)$ is determined by $A(X)$ & $B(X)$ again thr. 4 steps :

1. Setup of coordinates / basis (for $H^{even}(X) \xrightarrow[j]{\sim} H^{even}(Y)$)

all are invariant classes

$$s = s^0 \bar{T}_0 + \sum s^\epsilon \bar{T}_\epsilon + \sum s_\xi \bar{T}^\xi + s_0 \bar{T}^0 \in H(X) \cong jH(X)$$

$u = \sum_{\ell=1}^{\rho} u^\ell T_\ell$: Any $T_1, \cdots T_\rho$ basis of $\mathrm{Pic}(Y/\bar{X})_{\mathbb{Q}}$, $\perp$ $H_2(X)$

rel. divisors

Get $T^1, \cdots, T^\rho$ dual basis in $NE(Y/\bar{X})_{\mathbb{Q}}$

GW potential $F_0^Y(t)$, $t = s + u$ (only need small QH)

on $H^2(Y) \times K_{\mathbb{C}}^Y$ Kähler moduli space

For $Y \searrow X$, $K_{\mathbb{C}}^X \hookrightarrow K_{\mathbb{C}}^Y$ as a codim ρ boundary face

May choose T_ℓ's st. $b_{i\ell} = (C_i, T_\ell)$; $B = (b_{i\ell})$ rel. matrix for S_i's

2. Transversal Directions ($s = 0$)

Denote $f(q) = \sum_{d \in \mathbb{N}} q^d = \dfrac{q}{1-q}$

Proposition (multiple cover formula / Yukawa coupling)

$$C_{lmn} := \partial^3_{lmn} F^Y_0(u) = (T_l . T_m . T_n) + \sum_{i=1}^{k} b_{il}\, b_{im}\, b_{in}\, f\left(q^{[C_i]} e^{\sum_{p=1}^{p} b_{ip} u^p}\right)$$

The pole loci consist of k hyperplanes $E_i := \{ u \mid \sum_{p=1}^{p} b_{ip} u^p = 0 \}$

$$\sum_{p=1}^{p} b_{ip} u^p = \int_{C_i} u$$

3. Twisted A model on X

for $\gamma \in NE(Y)$: $\langle\ \rangle^X_\beta = \sum_{\gamma \mapsto \beta} \langle\ \rangle^Y_\gamma$: Trouble : Can't extract $\langle\ \rangle^Y_\gamma$!

$\gamma \mapsto \beta \neq 0$

Study GW theory on X "twisted by $S_1, \cdots, S_k$"

i.e. GW on $M := X \setminus \bigcup_{i=1}^{k} S_i$

Since topologically $M \sim Y \setminus \cup C_i \simeq U$, have $H_2(M) \simeq H_2(Y)$

For $C \hookrightarrow X$ with $\beta = [C]$ and $C \cap S_i = \phi$ $\forall i$, we call its class $\gamma \in H_2(M)$ a linking type of β wrt. $S_1, \cdots, S_k$

To define GWI of linking type γ, need $C \cap S_i = \phi$ $\forall i$ "virtually" among all rational curves

4. Rigidity: Can prove this using J. Li's degeneration formula in cycle form

Rmk: S_i can be repr. by Lagrangian spheres (Seidel, Donaldson)
can it be special Lagrangian?

still need to make the twisting $A(x) \rtimes B(x)$ effective (calculable)

Work in progress

Local model $\bar{X} = (2) \subset P^4$, $\mathcal{X}_t$: $uv - ws = tx^2$

$\psi: Y \longrightarrow \bar{X}$ is the blow up of Weil divisor $W \simeq P^2 = (u, w) \subset \bar{X}$

$$\begin{array}{ccccl} \Rightarrow Y & \overset{i}{\hookrightarrow} & \underset{[z_1 : z_2]}{P^1} \times P^4 = Bl_W P^4 & & \text{defined by } \frac{u}{w} = \frac{z_1}{z_2};\ \frac{s}{v} = \frac{z_1}{z_2} \\ & & \uparrow \sigma_1, \sigma_2 & & \text{ie. 2 sections } \sigma_1, \sigma_2 \text{ of } Q \simeq \mathcal{O}(1,1) \\ & & Q & & \end{array}$$

product formula + quantum Lefschetz $\Rightarrow$ $GW(Y)$

of course, $Y \simeq P_{P^1}(\mathcal{O}(-1)^2 \oplus \mathcal{O})$ is toric, and $GW(Y)$ is well known.

The point is the possibility to globalize the picture to general Calabi-Yau 3 folds.

$$\begin{tikzcd} H^2(X) \arrow[r] & H^2(Y) \arrow[d] \\ & \end{tikzcd}$$

$$0 \longrightarrow H^2(Y)/H^2(X) \xrightarrow{B} \mathbb{C}^k \xrightarrow{A^t} \mathbb{C}^M \cong V \longrightarrow 0$$

$B(X)$ provides PF for $\cup C_i$ and Weil divisors W_j, $1 \le j \le p$

On the level of space

$$\begin{tikzcd} \bar{X} & Y \arrow[l] \\ & \text{“} \cup C_i \text{”} \arrow[u] \end{tikzcd}$$

Expect a quantum Leray-Hirsch of the form

$$I^Y_\gamma = J^{\bar{X}}_\beta \cdot I^{Y/\bar{X}}_\gamma \qquad (\text{QLH})$$

$$J^{\bar{X}}_\beta \cong J^X_\beta \qquad \beta = \psi_*(\gamma)$$

Y = blow up of $\bar{X}$ along Weil divisors $W_1, \dots, W_p$ $\;\cong\; W \cup \{pts\}$

in each step $\quad \mathcal{E}^* := \mathcal{O}(-D_1) \oplus \mathcal{O}(-D_2) \twoheadrightarrow \mathcal{I}_Z \qquad Z = D_1 \cap D_2$

$$Y_1 = Bl_Z \bar{X} := \mathrm{Proj}_{\bar{X}} \bigoplus_{d=0}^{\infty} \mathcal{I}_Z^d \hookrightarrow \mathrm{Proj}\ \mathrm{Sym}\ \mathcal{E}^* = P_{\bar{X}}(\mathcal{E})$$

Y_1 may have extra blow up of Y at $\{pts\}$, which is OK.

$$0 \to S \to \pi^* \mathcal{E} \to Q \to 0$$

$$\begin{tikzcd} & & Q \arrow[d] \\ Y_1 \arrow[rr, hook, "i"] \arrow[drr, "\psi"'] & & P_{\bar{X}}(\mathcal{E}) \arrow[u, bend right, "\sigma"'] \arrow[d, "\pi"] \\ & & \bar{X} \end{tikzcd}$$

$s : \mathcal{E}^* \to \mathcal{I}_Z \to \mathcal{O}_{\bar{X}}$

$\pi^* s \in \Gamma(P_{\bar{X}}(\mathcal{E}), \pi^* \mathcal{E})$

$\sigma :=$ projection of $\pi^* s$ to Q

Let $p \in P_{\bar{X}}(\mathcal{E})$, $\sigma(p) = 0 \iff (\pi^* s)(p) = \underset{\text{fixed}}{s(\pi(p))} \in \underset{\text{varies without repetitions}}{S_p}$

when p moves along the $\mathbb{P}^1$ fiber at $\pi(p)$

$s(\pi(p)) \neq 0 \Rightarrow \exists!$ pt in fiber $\in S_p$

$s(\pi(p)) = 0$, i.e. $\pi(p) \in Z$, then the whole $\mathbb{P}^1$ fiber $\in S_p$

$\Rightarrow \quad \sigma^{-1}(0) = \underset{\text{main component}}{i(Y_1)} \cup \underset{\text{extra component \& extra blowup}}{\tilde{Z}}$

QLH for $P(\mathcal{E})$ + QHT for Q + deg. formula $\Rightarrow$ QLH for $Y \xrightarrow{\psi} \bar{X}$

END ※