Mathematics on Tori/3

CHIN-LUNG WANG
(NCU and NCTS)

July 14, 2004
Colloquium at Taiwan University

Part A: Classical Feature

1. Weierstrass's $\wp$ function
2. Classical Applications
3. Riemann's Theta Functions ϑ_i

Part B: Recent Feature (Selected)

4. Birational Geometry
5. Non-linear PDE
6. Arithmetic Geometry

1. Weierstrass's $\wp$ function

Lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

Torus $T = \mathbb{C}/L$. Genus $g(T) = 1$.

Analysis: Doubly periodic functions on $\mathbb{R}^2$.

Algebra: Elliptic Curves $y^2 = 4x^3 - g_2 x - g_3$.

Geometry: Flat tori. Curvature zero.

No holomorphic functions on T — **Liouville**.

Cauchy: no meromorphic with single pole:

$$\frac{1}{2\pi i}\int_C f(z)\,dz = 0.$$

Number of zeros = number of poles:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = 0.$$

Constraint $\sum_p \mathrm{ord}_p(f)\vec{p} = 0 \pmod{L}$ from

$$\frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}\,dz \in L.$$

Weierstrass: $\exists$ unique meromorphic function with double pole at 0.

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L^\times} \left(\frac{1}{(z-w)^2} - \frac{1}{\omega^2} \right).$$

$$\wp'(z) = -2 \sum_{\omega \in L} \frac{1}{(z-\omega)^3}.$$

By canceling poles of order 6,

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(L)\wp(z) - g_3(L).$$

Algebraic curve realization $\phi : T \to \mathbb{P}^2$ via

$$z \mapsto (x : y : 1) = (\wp(z) : \wp'(z) : 1).$$

2. Classical Applications

I. Calculus: non-integrability of elliptic integral as elementary functions.

Abel-Jacobi: extending the integral over $\mathbb{C}$,

$$\frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} = \frac{dx}{y} = \frac{d\wp(z)}{\wp'(z)} = dz.$$

The integral $z(x) = \int^x dx/y$ has a doubly periodic inverse function $x(z) = \wp(z)$, hence $z(x)$ can not be elementary.

II. Differential Equations:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

$$2\wp'\wp'' = 12\wp^2\wp' - g_2\wp',$$

$$2\wp'' = 12\wp^2 - g_2,$$

$$\wp''' = 12\wp\wp'.$$

$u(z,t) = \wp(z)$ is a solution of K-dV:

$$u_t = u_{zzz} - 12uu_z.$$

To get time-dependent solutions we need theta functions and to differentiate in $\tau = \omega_2/\omega_1$.

III. Algebra: Solving polynomial equations:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Abel, **Galois**: Not every polynomial equation can be solved by radicals.

Kronecker: Can solve polynomial equation of degree 5 in terms of radicals and $\wp(a; L)$.

Klein, **Jordan**, **Thomae**: Can solve all polynomial equations in terms of radicals and special values of (generalized) theta functions.

3. Riemann's Theta Function ϑ_i

$L = \mathbb{Z} + \mathbb{Z}\tau$, $\tau \in \mathrm{SL}(2, \mathbb{Z})\backslash\mathcal{H} = \mathcal{M}_1$.
Let $q = e^{\pi i \tau}$, $\tau = a + bi$, then $|q| = e^{-b} < 1$.

$$\vartheta_1(z;\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(n+\frac{1}{2})2\pi i z}.$$

It is an entire odd function with

$$\vartheta_1(z+1) = -\vartheta_1(z),$$
$$\vartheta_1(z+\tau) = -q^{-1} e^{-2\pi i z} \vartheta_1(z).$$

Heat equation:

$$\frac{\partial^2 \vartheta_1}{\partial z^2} = 4\pi i \frac{\partial \vartheta_1}{\partial \tau}.$$

Relations to Weierstrass theory:
$\wp'(z)$ is odd with zeros at $\omega_i/2$, $i = 1, 2, 3$.

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3),$$

with $e_i = \wp(\omega_i/2)$.
Let $\zeta(z) = -\int^z \wp(w)\,dw = 1/z + \cdots$, it is odd with quasi-periods

$$\eta_i = \zeta(z + \omega_i) - \zeta(z) = 2\zeta(\omega_i/2).$$

Let $\sigma(z) = e^{\int^z \zeta(w)\,dw}$. $\sigma(z) = z + \cdots$ is odd, entire with simple zeros at $z \in L$. Then

$$\sigma(z) = e^{\eta_1 z^2/2} \frac{\vartheta_1(z)}{\vartheta_1'(0)}.$$

Jacobi's imaginary transformation formula $\Rightarrow$ modularity for theta values: for $\tau\tau' = -1$,

$$\vartheta_1(z;\tau) = -i(i\tau')^{1/2} e^{\pi i \tau' z^2} \vartheta_1(z\tau';\tau').$$

Recall $\mathrm{SL}(2,\mathbb{Z}) = \langle S, T\rangle$ with $S\tau = -1/\tau \equiv \tau'$ and $T\tau = \tau + 1$.

Lemma 1 *For* $\hat{\tau} = ST^{-2}ST^{-1}\tau = \dfrac{\tau - 1}{2\tau - 1}$,

$$(\log\vartheta_1)_{\hat{\tau}}\left(\frac{1}{2};\hat{\tau}\right) = -(1-2\tau) + (1-2\tau)^2(\log\vartheta_1)_{\tau}\left(\frac{1}{2};\tau\right).$$

4. Birational Geometry

Let (X, h) be a complex hermitian n-manifold. Let $R = \nabla_h^2 \in \Lambda^2(\mathrm{End}(T_X))$ with **Chern** forms

$$c(X) = \det\left(I - \frac{1}{2\pi i}R\right) = 1 + c_1 + c_2 + \cdots$$

Let $\phi : Y \to X$ be a bi-moromorphic morphism. For $K_n \in \mathbb{C}[c_1, \cdots, c_n]$ a degree $2n$ form, we attach to it a "measure" $d\mu := K_n$.

Question: When do we have a CVF like

$$\int_X d\mu_X = \int_Y A(\phi)\, d\mu_Y,$$

with $A(\phi)$ depends only on $J\phi := \det D\phi$?

Hirzebruch: $\exists\, Q(x) = 1 + \cdots \in \mathbb{C}[[x]]$ s.t.

$$\int_X d\mu_X = \int_X \prod_{i=1}^{n} Q(x_i),$$

x_i are the Chern roots: $c(X) = \prod_{i=1}^{n}(1 + x_i)$.

Theorem 2 (W–, 2001) *Let $f(x) = x/Q(x)$. The CVF is valid "if and only if" there is a power series $A(x)$ such that*

$$\frac{1}{f(x)f(y)} = \frac{A(x)}{f(x)f(y-x)} + \frac{A(y)}{f(y)f(x-y)}.$$

Theorem 3 (W–, 2001) *The solutions of the functional equation are given by*

$$f(x) = e^{(k+\zeta(z))x} \frac{\sigma(x)\sigma(z)}{\sigma(x+z)},$$

and $A(x) = A(x, 2)$ *where*

$$A(x, r) = e^{-(r-1)(k+\zeta(z))x} \frac{\sigma(x+rz)\sigma(z)}{\sigma(x+z)\sigma(rz)}.$$

Thus they are parameterized by $\bar{\mathcal{M}}_{1,1} \times \mathbb{C}$*.*

Idea of proof: keep on differentiating, substitute A by $f^{(n)}$ and get some ODE. Solve the ODE by elliptic functions.

5. Non-linear PDE

This is a recent joint work with **C.-S. Lin** on the Mean Field Equation on a flat torus T:

$$\triangle u + 8\pi(e^u - 1) = 8\pi(\delta_0 - 1).$$

Theorem 4 *Existence of solutions correspond to existence of* **extra pair** *of critical points of Green's function.*

Let $G(z) = G(z, 0)$, then G is even, ∇G is odd and so $\nabla G(\omega_i/2) = 0$, $i = 1, 2, 3$.

Theorem 5 (Quiz: Who did this first?)

$$G(z,w) = -\frac{1}{2\pi}\log\left|\frac{\vartheta_1(z-w)}{\vartheta_1'(0)}\right| + \frac{1}{2b}(\text{Im}(z-w))^2.$$

Corollary 6 *For* $z = x + iy$, $\nabla G(z) = 0 \iff$

$$\frac{\partial G}{\partial z} \equiv \frac{-1}{4\pi}\left((\log\vartheta_1)_z + 2\pi i\frac{y}{b}\right) = 0.$$

Equivalently, $\zeta(t\omega_1 + s\omega_2) = t\eta_1 + s\eta_2$.

Theorem 7 *(a) For T a rectangle there are no extra critical points. (b)* For $\tau = (1+\sqrt{3}i)/2$ there are 5 critical points. (c)** For any flat tori, there are at most 5 critical points.*

One key point in the proof is to analyze the degeneracy condition of $\omega_i/2$ along the line $\Re\tau = 1/2$. Two inequalities are crucial:

$$(e_1 + \eta_1)_b > 0; \quad e_1 - 2\eta_1 > 0.$$

Let $A_n = n(n+1)/2$ and $r = e^{-2\pi b}$. Then

$$\begin{aligned}(e_1 + \eta_1)_b &= -4\pi(\log\vartheta_1(1/2))_{bb} \\ &= -f\sum_{n>m}(-1)^{A_n+A_m}(A_n - A_m)^2 r^{A_n+A_m} \\ &= f(r + 9r^3 - 4r^4 - 36r^6 + 25r^7 + \cdots).\end{aligned}$$

It is not hard to estimate that this is positive for $b \geq 1/2$ since then r is small.

What happens if $b \leq 1/2$ (and so r is large)?

6. Arithmetic Geometry

Goal: Solving polynomial equations in $\mathbb{Z}$.

Hasse-Minkowski: $f(x_1, x_2, \cdots, x_n) = 0$, f homogeneous of degree 2. Then $f = 0$ has $\mathbb{Z}$-solutions if and only if that
$f = 0 \pmod{p}$ has solutions for all prime p
and it has solutions in $\mathbb{R}$.

Selmer: Not true for cubic equations like

$$3X^3 + 4Y^3 + 5Z^3 = 0.$$

Motivic Approach: Let X be an elliptic curve over $\mathbb{Z}$, X_p be its reduction mod p. Consider the zeta function:

$$Z(X_p, t) = \sum_{k \geq 1} |X_p(\mathbb{F}_{p^k})| \frac{t^k}{k}$$
$$= \frac{f_p(t)}{(1-t)(1-pt)}.$$

Then $f_p(t) \in \mathbb{Z}[t]$, $\deg f(t) = 2$, and

$$f(\alpha) = 0 \Rightarrow |\alpha| = 1/\sqrt{p}.$$

This gives the L function as an **Euler** product:

$$L(X, s) = {\prod}'_p \frac{1}{f_p(p^{-s})}; \quad \mathrm{Re}\, s > \frac{3}{2}.$$

Wiles' Theorem. Let

$$\Lambda(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(X,s),$$

Hasse-Weil conjectured that $L(X,s)$ is entire and $\Lambda(2-s) = \pm\Lambda(s)$. Taniyama-Weil-Shimura conjectured that $L(X,s)$ is indeed a modular form. These are proved by A. Wiles.

Birch & Swinnerton-Dyer Conjecture: for r the Mordell-Weil rank of $X(\mathbb{Q})$,

$$L(X,s) = C(s-1)^r + \cdots$$

$$C = \frac{R}{2}|\mathrm{Sha}(X)||X(\mathbb{Q})_{\mathrm{tor}}|^{-2}\prod_p c_p \int_{X(\mathbb{R})} |dz|.$$

Arakelov: Let $S = \operatorname{Spec}\mathbb{Z}$ and $\pi : X \to S$ as an arithmetic surface. The genus of X can be any $g \geq 0$. Complete S by adding archimedean places as the S_∞. Here $S_\infty = \{\mathbb{Q} \hookrightarrow \mathbb{C}\}$.

X_p is the reduction of X mod p for $p \in S$.
X_∞ is simply the torus $X(\mathbb{C})$.

Let $P, Q \in X(\bar{\mathbb{Q}})$. They give rise to π-sections. Hence the intersection numbers

$$(P,Q)_{\mathsf{Ar}} := \sum_p (P,Q)_p + G(P,Q).$$

Key: $(D,D)_{\mathsf{Ar}}$ is defined by linear equivalence and is related to the height function.

Faltings: Arithmetic Riemann-Roch theorem, Hodge index theorem, Noether formula etc for curves X over $\mathbb{Q}$ of any genus $g \geq 0$. In particular, he defined the Arakelov divisor ω_X and proved $\omega_X^2 = (\omega_X, \omega_X)_{\mathsf{Ar}} \geq 0$.

Parshin (1986): (a) An upper bound for ω_X^2 in terms of K, g and places (primes) where X has bad reduction implies the Mordell conjecture. (b) the "Arithmetic **Miyaoka-Yau** inequality"

$$c_1^2 \leq 3c_2$$

implies the Fermat Last Theorem.