Geometric Transitions and Quantum Invariance

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Calabi–Yau manifolds

- A Calabi–Yau manifold X^n is a complex projective n-fold with $K_X \cong \mathscr{O}_X$ and $h^i(\mathscr{O}_X) = 0$ for $1 \leq i \leq n-1$.
- **Yau (1976):** Ricci flat metrics on X are in one to one correspondence with (J, ω) where J is a complex structure on X and $\omega \in H^{1,1}(X)$ is a Kähler class.
- **Bogomolov–Todorov–Tian (1987):** The deformation theory is unobstructed, namely the Kuranishi space $\mathscr{M}_X = Def(X)$ is smooth of dimension $h^{n-1,1}(X) = h^1(X, T_X)$.
- **Namilawa (1994):** BTT holds for Calabi-Yau 3-fold with at most terminal singularities.
  Local analytically $(p \in X) = cDV/\mu_r$ with
  $cDV = f(x,y,z) + tg(x,y,z,t)$ where f is an ADE equation.

Ried's fantesy: How to classify Calabi–Yau 3-folds?

- Finite topological type?
- Are Calabi-Yau 3-folds all "connected" through extremal transitions? Or even conifold (i.e. ODP) transitions?

$$\begin{array}{ccc} & & Y \\ & & \downarrow \psi \\ X = \mathfrak{X}_t & \overset{t \to 0}{\dashrightarrow} & \bar{X} \end{array}$$

where ψ is a projective crepant contraction and $\mathfrak{X}_t$ is a projective smoothing of $\bar{X} = \mathfrak{X}_0$. (Denote $Y \searrow X$, $X \nearrow Y$.)

- If $\bar{X}$ is a conifold with ODP $p_1, \cdots, p_k$, then Y contains k ϕ-exceptional curves $C_i \cong P^1$ with $N_{C_i/Y} \cong \mathscr{O}_{P^1}(-1)^{\oplus 2}$, X contains k vanishing spheres $S_i \cong S^3$ with $N_{S_i/X} \cong T^*S^3$:

$$\partial(S^3 \times D^3) = S^3 \times S^2 = \partial(D^4 \times S^2),$$

- Irreducible family via non-projective Calabi–Yau's??

Main examples

Up to date, there are more than 10^7 Calabi–Yau 3-folds found with different topological types!

- Complete intersections in toric varieties. E.g. $(5) \subset P^4$.
- **H. Clemens 1983:** Double solids. E.g. Branched double cover of P^3 along a degree 8 surface.
- **C. Schoen 1988:** Fiber product of elliptic surfaces

$$X = S_1 \times_{P^1} S_2,$$

  where $r_i : S_i \to P^1$ is a relatively minimal elliptic surface with section and without reduced fibers.
- The singular fibers are of type $I_n : t = xy$, $II : t = y^2 - x^3$, $III : t = x(y^2 - x)$, $IV : t = xy(x + y)$. If A_i is the critical value of r_i, then X is singular over $A_1 \cap A_2$. Any deformation of X is still of the form, hence smoothable.

Classical working problems

- **E. Viehweg 1990-97:** The moduli space $\mathscr{M}_h^c$ of polarized Calabi–Yau varieties with at most canonical singularities (with a fixed Hilbert polynomial h) is quasi-projective.
- **W- 1996:** The Weil–Petersson metric (for Ω a section of n forms)

$$\omega_{WP} := -\partial\bar{\partial} \log \tilde{Q}(\Omega, \bar{\Omega})$$

  has finite distance towards the boundary point of $\mathscr{M}_h$ which corresponds to CY with canonical singularities.
- **W- 2003:** MMP $\Rightarrow$ The converse holds for one dimensional moduli. Hence OK for Calabi–Yau 3-folds.
- **T.-J. Lee, W- 2013*:** The WP metric completion of $\mathscr{M}_h$ is precisely $\mathscr{M}_h^c$. (Small complex structure limits.)

- **Reid:** Is that possible to deform a terminal (or canonical) extremal transition $Y \searrow X$ into a conifold transition?
- **R. Friedman 1986:** The local contraction $(Y, C) \to (\bar{X}, p)$ can always be deformed into a ODP contraction $(Y', \coprod C_i) \to (\bar{X}', \{p_i\})$ with many ODP p_i's.
- Moreover, a ODP contraction $Y \to \bar{X}$ is globally smoothable if and only if there is a totally non-trivial relation $\sum a_i [C_i] = 0$ with $a_i \neq 0$ for all i.
- **Y. Namikawa 2002:** If $\bar{X} = S_1 \times_{\mathbb{P}^1} S_2$ has a type $III \times III$ singularity, then any extremal transition through $\bar{X}$ is not deformable into conifold trasnaitions!
- **S.-S. Wang 2012:** OK if we allow deformations, decompositions and flops. In fact 2 steps conifold transitions are enough for C. Schoen's examples.

Quantum aspects on projective conifold transitions

- The purpose of this talk is to give some observations on the quantum A and B models under a projective conifold transition $Y \searrow X$ of Calabi–Yau 3-folds. This is based on a joint project with H.-W. Lin and Y.-P. Lee.
- A model: Gromov–Witten theory.
- B model: Kodaira–Spencer theory (or VHS in the genus zero case).
- It is clear that $A(X) < A(Y)$ (Y has extremal rays) and $B(X) > B(Y)$ (X has vanishing cycles).
- But we expect that the full "TQFT" is "invariant" regardless the choices of CY's!
- Other aspects: Candelas, Strominger, Thomas–Yau, Tseng–Yau, Rong–Zhang, Xu, Lau (and many more ...).

Global constraint on conifold trsnaition $Y \searrow X$

- The Euler numbers satisfy

$$\chi(X) - k\chi(S^3) = \chi(Y) - k\chi(S^2).$$

That is, $\frac{1}{2}(h^3(X) - h^3(Y)) + (h^2(Y) - h^2(X)) = k$.

- Extremal transitions preserve $h^{3,0} = h^0(K)$, hence

$$\mu := \frac{1}{2}(h^3(X) - h^3(Y)) = h^{2,1}(X) - h^{2,1}(Y)$$

is the lose of complex moduli, and

$$\rho := h^2(Y) - h^2(X) = h^{1,1}(Y) - h^{1,1}(X)$$

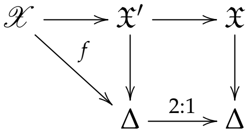
is the gain of Kähler moduli.

- The relation then reads as

$$\mu + \rho = k.$$

Factorization into two semi-stable reductions

- The transition $X \nearrow Y$ can be achieved as a composition of two semi-stable degenerations: $\mathscr{X} \to \Delta$ and $\mathscr{Y} \to \Delta$.
- The first one (complex degeneration) $f : \mathscr{X} \to \Delta$ is the semi-stable reduction



for $\mathfrak{X} \to \Delta$ obtained by a degree two base change $\mathfrak{X}' \to \Delta$ followed by the blow-up $\mathscr{X} \to \mathfrak{X}'$ of the 4D nodes

$$p_i' \in \mathfrak{X}', \quad i = 1, \ldots, k.$$

- The special fiber

$$\mathscr{X}_0 = X_0 \cup \coprod_{i=0}^{k} X_i$$

is a SNC divisor with

$$\tilde{\psi} : X_0 \cong \tilde{Y} \to \bar{X}$$

being the blow-up at all p_i's and $X_i = Q_i \cong Q \subset P^4$ is a quadric threefold for $i = 1, \ldots, k$.
- Let $X^{[j]}$ be the disjoint union of $j+1$ intersections from X_i's. Then $X^{[0]} = \tilde{Y} \coprod_i Q_i$ and $X^{[1]} = \coprod_i E_i$ where

$$E_i = \tilde{Y} \cap Q_i \cong P^1 \times P^1$$

are the $\tilde{\psi}$ exceptional divisors.

- The second one (Kähler degeneration) $g : \mathscr{Y} \to \Delta$ is simply the deformations to the normal cone

$$\mathscr{Y} = \mathrm{Bl}_{\coprod C_i \times \{0\}} Y \times \Delta \to \Delta.$$

- The special fiber

$$\mathscr{Y}_0 = Y_0 \cup \coprod_{i=1}^{k} Y_i$$

  with $\phi : Y_0 \cong \tilde{Y} \to Y$ being the blow-up along the curves C_i's and

$$Y_i = \tilde{E}_i \cong \tilde{E} = P_{P^1}(\mathscr{O}(-1)^2 \oplus \mathscr{O})$$

  for $i = 1, \ldots, k$.
- Non-trivial terms for $Y^{[j]}$ are $Y^{[0]} = \tilde{Y} \coprod_i \tilde{E}_i$ and $Y^{[1]} = \coprod_i E_i$ where $E_i = \tilde{Y} \cap \tilde{E}_i$ is the ∞ divisor of $\pi_i : \tilde{E}_i \to C_i \cong P^1$.

Limiting mixed Hodge theory

- Consider the period map $\phi(t,s)$ of a variation of Hodge structures $F^{\bullet}_{t,s}$ with unipotent monodromy T_i around D_i in the SNC divisor $D = \bigcup_{i=1}^{\mu} D_i$: Let $N_i = \log T_i$, $\Phi(z,s)$ its lifting with $t_i = e^{2\pi i z_i}$, and let $\Psi(z,s) := e^{-zN}\Phi(z,s)$ where $zN = \sum_{j=1}^{\mu} z_j N_j$. Then Ψ descends to $\psi : (t,s) \in \Delta^* \to D$:

$$\begin{array}{ccc}
\mathbb{H}^{\mu} \times \Delta^{h-\mu} & \xrightarrow{\quad \Phi \quad} & D \\
\downarrow & \overset{\psi}{\nearrow} & \downarrow \\
\Delta^* := (\Delta^{\times})^{\mu} \times \Delta^{h-\mu} & \xrightarrow{\quad \phi \quad} & D/\langle T_1, \cdots, T_{\mu} \rangle
\end{array}$$

- **W. Schmid's nilpotent orbit theorem 1971:** $\phi(t,s) = e^{zN}\psi(t,s)$ where ψ is holomorphic over Δ^h. $\psi(0,s) = F^{\bullet}_{\infty}(s)$ is called the limiting Hodge filtration. The nilpotent orbit $e^{zN}\psi(0,s)$ approximates ϕ "nicely".

- Let $\mathbf{a}(t,s)$ be a section of $\psi(t,s)^n$.

$$\mathbf{a}(t,s) = a_0(s) + \sum_{j=1}^{\mu} a_{1,j}(s)t_j + \cdots$$

with $a_0(s) \in F^n_\infty(s)$. Then $zN = \sum(\log t_j)N_j/2\pi i$,

$$\Omega(t,s) = e^{zN}\mathbf{a}(t,s) = e^{zN}a_0(s) + e^{zN}\sum a_{1,j}(s)t_j + \cdots$$

- In the case of conifold degenerations of Calabi–Yau 3-folds, $N_j a_0(s) = 0$ for all j and $N_i N_j = 0$ for any i, j. This follows from the one parameter case since $N = \sum n_j N_j$ along the curve $u \mapsto (u^{n_1}, \cdots, u^{n_\mu}, s)$ for any fixed s.
- We first consider the one parameter case hence

$$\Omega(t) = a_0 + \frac{t \log t}{2\pi i} N a_1 + \cdots.$$

$F^\bullet_\infty$ and W_N defines a MHS. We will see that $N^2 = 0$ soon.

- Now we compare the MHS on $H(\mathscr{X}_0)$, computed from $E_1^{p,q}(\mathscr{X}_0) = H^q(X^{[p]})$ with Čech $\delta : H^q(X^{[p]}) \to H^q(X^{[p+1]})$, and the limiting MHS on $H(X)$ (also $H(\mathscr{Y}_0)$ and $H(Y)$):
- The Clemens–Schmid exact sequences for MHS's are

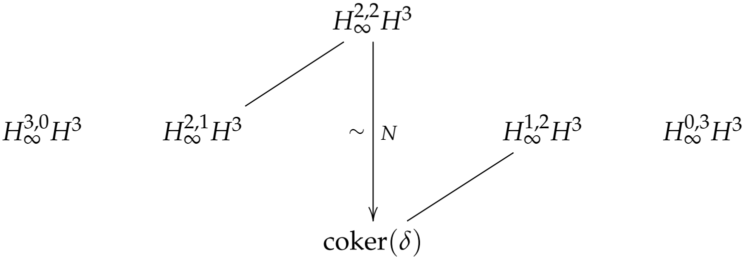
$$0 \to H^3(\mathscr{X}_0) \to H^3(X) \xrightarrow{N} H^3(X) \to H_3(\mathscr{X}_0) \to 0,$$
$$0 \to H^0(X) \to H_6(\mathscr{X}_0) \to H^2(\mathscr{X}_0) \to H^2(X) \xrightarrow{N} 0,$$
$$0 \to H^3(\mathscr{Y}_0) \to H^3(Y) \xrightarrow{N} 0,$$
$$0 \to H^0(Y) \to H_6(\mathscr{Y}_0) \to H^2(\mathscr{Y}_0) \to H^2(Y) \xrightarrow{N} 0,$$

  where N is trivial for $\mathscr{Y} \to \Delta$.
- Since $H^2(\mathscr{X}_0)$ is of weight 2, N on $H^2(X)$ is also trivial and the Hodge structure does not degenerate at all.

- Let $K = \ker(N : H^3(X) \to H^3(X)) \cong H^3(\mathscr{X}_0)$. Then

$$K \cong H^3(Y) \oplus \operatorname{coker}(\delta).$$

- From the the limiting Hodge diamond,



we conclude that $G_3^W H^3(X) \cong H^3(Y)$ and

$$\mu = h^{2,2}_\infty H^3 = h^{1,1}_\infty H^3 = \dim \operatorname{coker}(\delta).$$

- **Lemma.** $V^* \cong H^{2,2}_\infty H^3$ and $V \cong H^{1,1}_\infty H^3$.
- *Proof:* For any 3-fold isolated singularities,

$$0 \to V \to H_3(X) \to H_3(\bar{X}) \to 0$$

is exact. Dually $0 \to H^3(\bar{X}) \to H^3(X) \to V^* \to 0$.
- The invariant cycle theorem (c.f. BBD) implies that $H^3(\bar{X}) \cong \ker N = K \cong H^3(\mathscr{X}_0)$. Hence

$$V^* \cong H^{2,2}_\infty H^3 = F^2_\infty G^W_4 H^3(X).$$

The non-degeneracy of $Q(N\alpha, \beta)$ on $G^W_4 H^3(X)$ implies that

$$H^{1,1}_\infty H^3 = NH^{2,2}_\infty H^3 \cong (H^{2,2}_\infty H^3)^* \cong V^{**} \cong V.$$

- **Theorem (Basic exact sequence)**

  The group of vanishing S^2 cycles on Y and the group of vanishing S^3 cycles on X are linked by the weight 2 exact sequence

$$0 \to H^2(Y)/H^2(X) \xrightarrow{B} \bigoplus_{i=1}^{k} H^2(E_i)/H^2(Q_i) \xrightarrow{A^t} K/H^3(Y) \cong V \to 0.$$

  Here $A \in M_{k\times\mu}(\mathbb{Z})$ is the relation matrix for C_i's and $B \in M_{k\times\rho}(\mathbb{Z})$ is the relation matrix for S_i's. In particular

$$B = \ker A^t \quad \textit{and} \quad A = \ker B^t.$$

  - **Remark:** This sequence in fact splits: $0 \to \mathbb{Z}^\rho \to \mathbb{Z}^k \to \mathbb{Z}^\mu \to 0$. We eventually want to have a $\mathscr{D}$ module version (non-split) of this.

A key construction

- Consider the topological construction: For any non-trivial relation $\sum_{i=1}^{k} a_i[C_i] = 0$, there is a 3-chain W in Y with

$$\partial W = \sum_{i=1}^{k} a_i C_i.$$

- Under $\psi : Y \to \bar{X}$, C_i collapses to the node p_i hence $\bar{W} := \psi_* W \in H_3(\bar{X}, \mathbb{Z})$.
- As in Lemma, $\bar{W}$ deformes (lifts) to $\gamma \in H_3(X, \mathbb{Z})$ in nearby fibers. Using the intersection pairing, we get

$$PD(\gamma) \in H^3(X, \mathbb{Z}).$$

Restricting to the vanishing cycle space V, $PD(\gamma) \in V^*$.

- In the proof we establish the correspondence for each column vector $A_j = (a_{1j}, \cdots, a_{kj})^t$ with the element $PD(\gamma_j) \in V^*$, $1 \leq j \leq \mu$, characterized by

$$a_{ij} = (\gamma_j . S_i).$$

- Dually, we denote by $T_1, \cdots, T_\rho \in H^2(Y)/\text{tor}$ those divisors which form an integral basis of the lattice in $H^2(Y)$ dual (othogonal) to $H_2(X) \subset H_2(Y)$. In particular they form an integral bases of $H^2(Y)/H^2(X)$.
- Notice that we may choose T_l's, $l = 1, \ldots, \rho$, such that T_l corresponds to the l-th column vector of the matrix B via

$$b_{il} = (C_i . T_l).$$

The implication $(A(X), B(X)) \Rightarrow (A(Y), B(Y))$

Gromov–Witten and Dubrovin connections

Using the degeneration formula, we may relate the GW theory on X with that on Y by way of $\tilde{Y}$.

- For $\beta \in NE(X)\setminus\{0\}$ and $\vec{a} \in H_{inv}(X)^{\oplus n}$,

$$\langle \vec{a} \rangle^X_{g,n,\beta} = \sum_{\psi_*(\gamma)=\beta} \langle j(\vec{a}) \rangle^Y_{g,n,\gamma}$$

  where $j : H_{inv}(X) \to H(Y)$ is defined by $j(a) = \phi_*(a_0)$ with $(a_i)_{i=0}^k \in H(\tilde{Y} \coprod Q_i)$ being the admissible lifting of a with $a_i = 0$ for all $i \neq 0$. The sum is indeed finite!

- For 3-fold conifold transitions and for even dimensional classes it was first derived by Li–Ruan using symplectic glueing formula and later reinterpreted by Liu–Yau using Jun Li's algebraic degeneration formula.

- Let $s = \sum_\epsilon s^\epsilon \bar{T}_\epsilon \in H^2(X)$ where $\bar{T}_\epsilon$'s is a basis of $H^2(X)$. The pre-potential function is given by

$$F_0^X(s) = \sum_{n=0}^{\infty} \sum_{\beta \in NE(X)} \langle s^n \rangle_{0,n,\beta} \frac{q^\beta}{n!} = \frac{s^3}{3!} + \sum_{\beta \neq 0} n_\beta^X q^\beta e^{(\beta.s)},$$

  where $n_\beta^X = \langle \rangle_{0,0,\beta}^X$, with formal variables q^β's.
- It is a function in the Kähler moduli via $q^\beta = \exp 2\pi i(\beta.\omega)$, $\omega = B + iH$ in the complexified Kähler cone $\mathcal{K}_{\mathbb{C}}^X$ of X.
- Strictly speaking we need to consider $s \in H^{ev}(X)$. This will only change the topological part $s^3/3!$ with

$$s = s^0 \bar{T}_0 + \sum_\epsilon s^\epsilon \bar{T}_\epsilon + \sum_\zeta s_\zeta \bar{T}^\zeta + s_0 \bar{T}^0.$$

  Notice: We use Greek indices for variables from $H(X)$.

- Similarly we have $F_0^Y(t)$ on $H^2(Y) \times \mathcal{K}_{\mathbb{C}}^Y$. Here

$$t = s + u$$

with respect to $H^2(Y) = jH^2(X) \oplus \bigoplus_{l=1}^{\rho} \mathbb{Z} T_l$ and write

$$u = \sum_{l=1}^{\rho} u^l T_l.$$

- For $C \cong P^1$ with twisted bundle $N = \mathscr{O}_{P^1}(-1)^{\oplus 2}$,

$$E_0^C(t) = \sum_{d \in \mathbb{N}} n_d^N q^{d[C]} e^{d(C.t)} = \sum_{d \in \mathbb{N}} \frac{1}{d^3} q^{d[C]} e^{d(C.t)}.$$

- We also consider the total (global) extremal function

$$E_0^Y(t) := \frac{t^3}{3!} + \sum_{i=1}^{k} E_0^{C_i}(t).$$

where $E_0^{C_i}(t)$ depends only on u.

- Hence a splitting of variables

$$F_0^Y(s+u) = F_0^X(s) + E_0^Y(u) + \frac{1}{3!}((s+u)^3 - s^3 - u^3).$$

  The structural coefficients for $QH^{ev}(Y)$ are $C_{PQR} = \partial^3_{PQR} F_0^Y$.
- The part $F_0^X(s)$ simply comes from $QH^{ev}(X)$.
- For the part $E_0^Y(u)$,

$$\begin{aligned} C_{lmn} &= (T_l.T_m.T_n) + \sum_{i=1}^{k} \sum_{d\in\mathbb{N}} (C_i.T_l)(C_i.T_m)(C_i.T_n)\, q^{d[C_i]} e^{d(C_i.u)} \\ &= (T_l.T_m.T_n) + \sum_{i=1}^{k} b_{il} b_{im} b_{in} \mathbf{f}(q^{[C_i]} \exp \textstyle\sum_{p=1}^{\rho} b_{ip} u^p). \end{aligned}$$

- Here

$$\mathbf{f}(q) = \sum_{d\in\mathbb{N}} q^d = \frac{q}{1-q} = -1 + \frac{-1}{q-1}.$$

- The degeneration loci $E = \bigcup_{i=1}^{k} E_i$ of the GW theory consists of the k hyperplanes defined by

$$E_i := \left\{ u \mid w_i := \sum_{p=1}^{\rho} b_{ip} u^p = 0 \right\}.$$

Whenever $\rho > 1$, E is not a normal crossing divisor.
- The Dubrovin connection on $TH^{ev}(Y)$

$$\nabla^z = d - \frac{1}{z} \sum_P dt^P \otimes T_P *$$

"restricts" to the Dubrovin connection on $TH^{ev}(X)$.
- For the other part with basis T_l's and T^l's, we have

$$z \nabla^z_{\partial_l} T^m = -\delta_{lm} T^0,$$
$$z \nabla^z_{\partial_l} T_m = -\sum_{n=1}^{\rho} C_{lmn}(u) T^n - \sum_{\epsilon} C_{lm\epsilon} \bar{T}^{\epsilon},$$
$$z \nabla^z_{\partial_\epsilon} T_m = -\sum_{n=1}^{\rho} C_{\epsilon mn} T^n.$$

- E.g. the nilpotent monodromy $N^{(i)}$ along E_i is given by

$$N^{(i)}_{mn} = \frac{2\pi i}{z} b_{im} b_{in}.$$

- Unfortunately, for $\beta \neq 0$, in the finite sum

$$\langle - \rangle^X_\beta = \sum_{d_i} \langle - \rangle^Y_{j(\beta)+\sum_{i=1}^k d_i [C_i]}$$

  we still need to extract the individual term to determine $QH(Y)$ completely.
- WDVV equations can help to determine the off diagonal constants $C_{\epsilon mn}$'s, but give no further constraints.
- Indeed, the term with $\gamma = j(\beta) + \sum_{i=1}^k d_i [C_i]$ corresponds to those $C \subset X$, $[C] = \beta$, and the linking number $L(C, S_i)$ of C with S_i is d_i for $i = 1, \ldots, k$.

The implication $(A(Y), B(Y)) \Rightarrow (A(X), B(X))$

Periods and Gauss–Manin connections

- Recall ∇^{GM} on $\mathscr{H}^k = R^k f_* \mathbb{C} \otimes \mathscr{O}_S \to S$ for a smooth family $f : \mathscr{X} \to S$ is a flat connection with flat sections $R^k f_* \mathbb{C}$.
- Let $\delta_i \in H_k(X, \mathbb{Z})/\text{tor}$ be a homology basis for a fixed refernce fiber $X = \mathscr{X}_{s_0}$, with dual basis $\delta_i^* \in H^k(X, \mathbb{Z})$. Then δ_i^* can be extended to be (multi-valued) flat sections in $R^k f_* \mathbb{Z}$. For $\eta \in \Gamma(S, \mathscr{H}^k)$, we may write

$$\eta = \sum_i \delta_i^* \int_{\delta_i} \eta,$$

  with coefficients being the "multi-valued" period integrals.
- Let (s_j) be a local coordinates system in S. Then

$$\nabla^{GM}_{\partial/\partial s_j} \eta = \sum_i \delta_i^* \int_{\delta_i} \frac{\partial}{\partial s_j} \eta.$$

- When $f : \mathscr{X} \to S$ contains singular fibers, ∇^{GM} admits a logarithmic extension to the boundary.
- We need to investigate the local complex moduli space of X towards the conifold degeneration boundary D:

$$\begin{array}{ccc} & & \mathscr{M}_Y \\ & & \downarrow \pi \\ \mathscr{M}_X & \longrightarrow & \widetilde{\mathscr{M}}_{\bar{X}} \supset D \supset \pi(\mathscr{M}_Y) \end{array}$$

- By the BBT unobstructedness theorem, periods of vanishing cycles give rise to a natural coordinates system of the deformations of X in the transversal directions towards $D \ni [\bar{X}]$ with the same singularity type.
- The "invariant periods" then lift to the small resolution Y to give rise to the periods on Y.

- Let $A = (a_{ij}) \in M_{k\times\mu}(\mathbb{Z})$ be the relation matrix for C_i's. Recall the basis $\{PD(\gamma_j)\}_{j=1}^{\mu}$ of vanishing cocycles V^*:

$$PD(\gamma_j)([S_i]) \equiv (\gamma_j.S_i) := a_{ij}, \quad 1 \le j \le \mu.$$

  We may choose $\gamma_j \in H_3(X)$ so that $\gamma_j \in H_3(Y)^{\perp}$.
- Vanishing cycles: Let $\Gamma_j \in V$ be the dual basis, $(\Gamma_j.\gamma_l) = \delta_{jl}$.
- We may construct a symplectic basis of $H_3(X,\mathbb{Z})$:

$$\alpha_0, \alpha_1, \cdots, \alpha_h, \beta_0, \beta_1, \cdots, \beta_h, \quad (\alpha_j.\beta_k) = \delta_{jk},$$

  where $h = h^{2,1}(X)$, with $\alpha_j = \Gamma_j$ for $1 \le j \le \mu$.
- Then any $\eta \in H^3(X,\mathbb{C}) \cong \mathbb{C}^{2(h+1)}$ is identified with

$$\eta = \sum_{i=0}^{h} \alpha_i^* \int_{\alpha_i} \eta + \beta_i^* \int_{\beta_i} \eta.$$

- The symplectic basis property implies that

$$\alpha_i^*(\Gamma) = (\Gamma.\beta_i) \qquad \beta_i^*(\Gamma) = -(\Gamma.\alpha_i) = (\alpha_i.\Gamma).$$

- This leads to the important observation that we may modify γ_j by vanishing cycles to get

$$\gamma_j = \beta_j.$$

  So, $(\gamma_j.\gamma_l) = 0$ for $1 \le j, l \le \mu$ and $(\alpha_j^*.S_i) = (S_i.\beta_j) = -a_{ij}$.
- Bryant–Griffiths: $w_i = \int_{\alpha_i} \Omega$ form the coordinates of the image of the period map in $P(H^3) \cong P^{2h-1}$ as a Legendre submanifold of the holomorphic contact structure.
- By the flatness of ∇^{GM}, there is a holomorphic pre-potential function $u(w_0, \cdots, w_h)$ such that

$$u_i = \frac{\partial u}{\partial w_i} = \int_{\beta_i} \Omega,$$

  and hence

$$\Omega = \sum_{i=0}^{h} w_i \alpha_i^* + u_i \beta_i^*.$$

- In particular,

$$\partial_i\Omega = \alpha_i^* + \sum_{j=1}^{h} u_{ij}\beta_j^*, \qquad \partial_{ij}^2\Omega = \sum_{k=1}^{h} u_{ijk}\beta_k^*.$$

- By the Griffiths transversality, $\partial_i\Omega \in F^2$, $\partial_{ij}\Omega \in F^1$, and all are orthogonal to F^3. Hence we have the cubic form

$$u_{ijk} = (\partial_k\Omega.\partial_{ij}^2\Omega) = \partial_k(\Omega.\partial_{ij}^2\Omega) - (\Omega.\partial_{ijk}^3\Omega) = -(\Omega.\partial_{ijk}^3\Omega).$$

  This is known as the Yukawa coupling.
- We will write down the extension of the Yukawa coupling across the degenerate loci $D \subset \mathscr{M}_{\bar{X}}$.

- Recall Friedman's result on partial smoothing of ODP's in the following form: Let $A = [A^1, \cdots, A^\mu]$ be the relation matrix. For any $r \in \mathbb{C}^\mu$, the relation vector

$$A_r := \sum_{j=1}^{\mu} r_j A^j$$

  gives rise to a (germ of) partial smoothing of those ODP's $p_i \in \bar{X}$ with $A_{r,i} \neq 0$.
- Thus for $1 \leq i \leq k$, the linear equation

$$w_i := \pi_i(A_r) = r_1 a_{i1} + \cdots + r_\mu a_{i\mu} = 0$$

  defines a codimension one hyperplane $D^i \subset \mathbb{C}^\mu$.
- $D = \bigcup_{i=1}^k D^i \subset \mathbb{C}^\mu$ is NOT a SNC.

- Now the small resolution $\psi : Y \to \bar{X}$ leads to an embedding $\mathcal{M}_Y \subset \mathcal{M}_{\bar{X}}$ of co-dimension μ. As germs of analytic spaces we thus have

$$\mathcal{M}_{\bar{X}} \cong \Delta^{\mu} \times \mathcal{M}_Y \ni (r, s).$$

- Along each hyperplane D^i there is a monodromy operator T_i with associated nilpotent monodromy $N_i = \log T_i$.
- A degeneration from X to X_i with $[X_i] \in D^i$ a general point ($\notin D^{i_1}$ with $i_1 \neq i$) contains only one vanishing cycle

$$[S_i^3] \mapsto p_i.$$

- The Picard–Lefschetz formula says that for any $\sigma \in H^3(X)$,

$$N_i \sigma = (\sigma . PD([S_i^3])) PD([S_i^3]).$$

- If a period on a vanishing cycle Γ is single valued then it admits continuous extensions to Δ^h, hence is holomorphic on Δ^h. This is equivalent to that for all $i = 1, \dots, k$

$$\int_\Gamma N_i \mathbf{a}(r,s) = 0.$$

- By a holomorphic change of coordinates, and by shirking the neighborhood if necessary, we may assume that $\theta_j(r,s) := \int_{\Gamma_j} \Omega(r,s) = r_j$ for $1 \leq j \leq \mu$. In particular,

$$\Omega(r,s) \equiv \mathbf{a}(r,s) \equiv \sum_{j=1}^{\mu} \Gamma_j^* r_j \pmod{V^\perp}.$$

Proposition

In such parameters $\Omega(r,s)$ takes a simple form

$$\Omega = a_0(s) + \sum_{j=1}^{\mu} \Gamma_j^* r_j + \text{h.o.t.} - \sum_{i=1}^{k} \frac{w_i \log w_i}{2\pi i} PD([S_i]).$$

Here h.o.t. denotes terms in $V^\perp$ which are at least quadratic in $r_1, \cdots, r_\mu$.

- Indeed, by embedded resolution and the nilpotent orbit theorem we have

$$\Omega = a_0(s) + \sum_{j=1}^{\mu} \Gamma_j^* r_j + \text{h.o.t.} + \sum_{i=1}^{k} \sum_{j=1}^{\mu} \frac{\log w_i}{2\pi i} N_i \Gamma_j^* r_j.$$

Then

$$\sum_{j=1}^{\mu} N_i \Gamma_j^* r_j = -\sum_{j=1}^{\mu} a_{ij} PD([S_i]) r_j = w_i PD([S_i]).$$

- Since $\Omega(s) = a_0(s)$ for $s \in \mathscr{M}_Y$,

$$u_p(r,s) = \int_{\beta_p} \Omega = u_p(s) + \text{h.o.t.} - \sum_{i=1}^{k} \frac{w_i \log w_i}{2\pi i} \int_{\beta_p} PD([S_i]).$$

- For $1 \leq p \leq \mu$ we get

$$u_p(r,s) = \int_{\beta_p} \Omega = u_p(s) + \text{h.o.t.} + \sum_{i=1}^{k} \frac{w_i \log w_i}{2\pi i} a_{ip}.$$

- Otherwise we get simply $u_p(r,s) = u_p(s) + \text{h.o.t.}$.
- The asymptotic of the Yukawa coupling is determined:

$$u_{pm} = \text{h.o.t} + \sum_{i=1}^{k} \frac{\log w_i + 1}{2\pi i} a_{ip} a_{im},$$

$$u_{pmn} = \text{h.o.t.} + \sum_{i=1}^{k} \frac{1}{2\pi i} \frac{1}{w_i} a_{ip} a_{im} a_{in}.$$

Conclusion:

- We still don't know how to connect two Calabi–Yau 3-folds of different topological types through extremal transitions.
- If there is indeed an extremal transition $Y \searrow X$, then it is reasonable to expect that it can be decomposed/deformed into conifold transitions up to flops.
- For a conifold transition $X \nearrow Y$, $(A(X), B(X))$ determines $(A(Y), B(Y))$ up to knowledge of linking numbers $L(C, S_i)$. While $\mathscr{M}_Y \subset \mathscr{M}_X$, $A(Y)$ is only partially determined by $A(X)$ and the relation matrix B of vanishing spheres S_i's.
- $(A(Y), B(Y))$ determines $(A(X), B(X))$ up to regular terms of the Gauss–Manin connection on $\mathscr{M}_{\bar{X}}$. ∇^{GM} on $\mathscr{M}_Y$ gives the boundary Yukawa coupling, the log part is determined by the relation matrix A of the extremal curves C_i's.