The Role of Chern Classes in Birational Geometry

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Outline

1. First Chern Class and the Minimal Model Theory
2. c_1-Equivalence, Volume Equivalence and Motivic Theory
3. Chern Numbers, Complex Cobordism and the Weak Decomposition Theorem
4. Invariance of Quantum Ring under Simple Ordinary Flops

1. First Chern Class and the Minimal Model Theory

1.1 Chern Classes

Axioms:

1. For $E \to X$ a complex vector bundle of rank r,
   $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)$, $c_i(E) \in H^{2i}(X, \mathbb{Z})$
2. Naturality. For $f : Y \to X$, $c(f^*E) = f^*c(E)$.
3. Whitney Sum. For $0 \to E \to F \to G \to 0$, $c(F) = c(E).c(G)$.
4. Normalization. $c(\mathcal{O}_{\mathbb{CP}^1}(-1)) = 1 - h$.

The top Chern class $c_r(E)$ is called the Euler class $e(E)$. Its Poincaré dual is the zero locus (σ) for a generic section $\sigma : X \to E$.

Chern-Weil Theory.

Given a connection $\nabla : A^0(E) \to A^1(E)$ with curvature $R = \nabla^2 : A^0(E) \to A^2(E)$, it is known that $R \in A^2(\mathrm{End}\, E)$.

$$c(E, \nabla) := \det\left(1 + \frac{\sqrt{-1}}{2\pi} R\right) = 1 + c_1(E, \nabla) + c_2(E, \nabla) + \cdots + c_r(E, \nabla).$$

$[c_i(E, \nabla)] = c_i(E)$ via de Rham isomorphism.

When X is complex and (E, h) is holomorphic/hermitian, there is a unique hermitian connection ∇. Write $h = (h(\sigma_i, \sigma_j))_{i,j=1}^r$, then

$$R = \bar\partial(h^{-1}\partial h) \in A^{1,1}(\mathrm{End}\, E); \quad \mathrm{Tr}\, R = \bar\partial\partial \log\det h.$$

In particular, for $E = TX$, the Ricci form $Ric = -\partial\bar\partial \log\det g$. Also $[\frac{\sqrt{-1}}{2\pi} Ric] = c_1(X) = -c_1(K_X)$. Where $K_X = \Lambda^{\dim X} T^*X$.

Chern Classes in Algebraic Geometry

For $E \to X$ algebraic, say X is smooth, we have $c_i(E) \in A^i(X)$.
Example 1. Intersection product (Fulton):

$$\begin{array}{ccc} f^{-1}(X) = W & \overset{j}{\hookrightarrow} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \overset{i}{\hookrightarrow} & Y \end{array}$$

where i is a regular imbedding of codimension d with normal bundle $N_{X/Y}$. V a k dimensional scheme, $N := g^* N_{X/Y} \to W$,

$$X.V := \{c(N) \cap s(W, V)\}_{k-d} \in A_{k-d}(W).$$

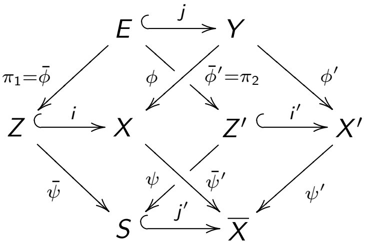
Example 2. $c_1 \equiv -K$ **and birational geometry.**

Two smooth projective varieties X and X' are birational if they have a common Zariski open set $f : U \xrightarrow{\sim} U'$. I.e. They are two different compatifications. The exceptional loci of X and X' can be compared using $c_1(X)$ and $c_1(X')$!

Ordinary (r, r')**-flips.**

$F \to S$, $F' \to S$: two vector bundles of rank $r+1$, $r'+1$.
$\bar{\psi} : Z := \mathbb{P}_S(F) \to S$, $\bar{\psi}' : Z' := \mathbb{P}_S(F') \to S$.
$E := \mathbb{P}_S(F) \times_S \mathbb{P}_S(F')$ with $\bar{\phi} : E \to Z$ and $\bar{\phi}' : E \to Z'$.
Let Y be the total space of $N := \bar{\phi}^* \mathcal{O}_Z(-1) \otimes \bar{\phi}'^* \mathcal{O}_{Z'}(-1)$
$E =$ zero section, $N_{E/Y} = N$.

$$X = \text{the total space of } \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^* F' = N_{Z/X},$$
$$X' = \text{the total space of } \mathcal{O}_{\mathbb{P}_S(F')}(-1) \otimes \bar{\psi}'^* F = N_{Z'/X'}.$$



$(F, F') \sim (F_1, F_1') \Leftrightarrow (F_1, F_1') = (F \otimes L, F' \otimes L^*)$ for $L \in \operatorname{Pic} S$.

$K_Y = \phi^* K_X + r' E = \phi'^* K_X' + rE$. So $X \geq_K X' \Leftrightarrow r \geq r'$.

An (r, r) flip is called an (ordinary) $\mathbb{P}^r$ flop.

1.2 K-Partial Order and Minimal Model Theory

Minimal Model Program: A normal variety X is terminal if K_X is $\mathbb{Q}$-Cartier and for some resolution $\phi : Y \to X$, one has $K_Y = \phi^* K_X + \sum a_i E_i$ with $a_i > 0$.

Theorem (Mori, Kawamata, Shokurov)

Let X be terminal. If K_X is not nef, then each extremal ray $R \in \overline{NE}_{K<0}$ is spanned by a rational curve. There exists contraction $\psi_R : X \to \bar{X}$ such that $\psi_R(C') = pt \Leftrightarrow [C'] \in R$.

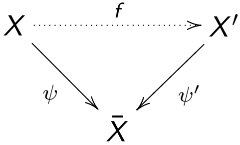
One ends up with 3 possibilities on $\bar{X}$:

1. $\dim \bar{X} < \dim X$, so $X \to \bar{X}$ is a fiber space, OK.
2. ψ_R is divisorial, i.e. $\dim \mathrm{Exc}(\phi_R) = n - 1$, OK.
3. ψ_R is small, i.e. $\dim \mathrm{Exc}(\phi_R) < n - 1$. $\bar{X}$ is not $\mathbb{Q}$-Gorenstein!

X is a **minimal model** if it is terminal and K_X is nef.

Three Dimensional Flips/Flops

A $(K_X + D)$ log-flip of a log-extremal contraction ψ is a diagram



st. f is isomorphic in codimension one and $K_{X'} + D'$ is ψ'-ample.

$D = 0$ is called a flip. K_X is ψ-trivial is called a D-flop.

Theorem (Mori 1988, Kollár-Mori 1992, Shokurov 2002)

3D log-flips exist in families. Also 3D birational $\mathbb{Q}$-factorial minimal models are related by a sequence of flops.

Summary of 3D Mori Theory

∞. The MMP ends up with a $\mathbb{Q}$-factorial minimal model.

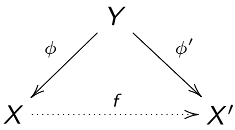
3. The minimal models are not unique, but any two are related by a sequence of flops. Moreover, flops are classified.
2. $Def(X) \cong Def(X')$ canonically.
1. $H^*(X) \cong H^*(X')$, $IH^*(X) \cong IH^*(X')$ which are compatible with the mix (pure) Hodge structures.
0. X' has the same singularity type as X.

What should one expect in HD?

∞ is infinitely hard. But **1**, **2** and **3** do not depend on it. Even in 3D, the ring structures in **1** is usually different. **0** is wrong.

K-Partial Order and K-Equivalence

Two $\mathbb{Q}$-Gorenstein varieties X and X' has $X \geq_K X'$ if



such that $\phi^* K_X \geq \phi'^* K_{X'}$. Examples are divisorial contradictions and flips. Flops satisfy $X =_K X'$.

Theorem

1. *If X and X' are birational terminal varieties such that $K_{X'}$ is nef along the exceptional loci then $X \geq_K X'$.*
2. *If $X =_K X'$ and $\dim X = 3$, then $f : X \dashrightarrow X'$ can be decomposed into a sequence of flops.*

Application to the filling in problem. Let $\mathcal{X} \to \Delta$ be a projective smoothing of a singular minimal Gorenstein 3-fold $\mathcal{X}_0$. Then $\mathcal{X} \to \Delta$ is not birational to a projective smooth family $\mathcal{X}' \to \Delta$ up to any base change.

Sketch. Inversion of adjunction $\Rightarrow \mathcal{X}$ is terminal Gorenstein.
So $\mathcal{X} =_K \mathcal{X}'$ and $\mathcal{X}_0$ is birational to $\mathcal{X}'_0$.
$\mathcal{X}_0$ is not $\mathbb{Q}$-factorial since it is singular.
Consider the $\mathbb{Q}$-factorialization $X \to \mathcal{X}_0$. Then $X \sim \mathcal{X}_0 \sim \mathcal{X}'_0$, hence X is smooth and $H^*(X) \cong H^*(\mathcal{X}'_0) \cong H^*(\mathcal{X}'_t) \cong H^*(\mathcal{X}_t)$.
If $\mathcal{X}_0$ has only ODP, we get a contradiction by formula for b_i.
For general cDV, we use symplectic deformations to reduce it to the ODP case.

$$\begin{array}{ccccc} X & & & & \\ \downarrow{\scriptstyle \phi} & & & & \\ \mathcal{X}_0 & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}_t. \end{array}$$

Main Conjectures (on c_1-equivalent manifolds)

Let $X =_K X'$ via $f : X \dashrightarrow X'$.

I. There exists a canonical correspondence $\mathcal{F} = \bar{\Gamma}_f + \sum_i \mathcal{F}_i \subset A^n(X \times X')$, with $\mathcal{F}_i$ being degenerate correspondences, which defines an isomorphism on Chow motives. E.g. $\mathbb{Q}$-Hodge structures.

II. $\mathrm{Def}(X) \cong \mathrm{Def}(X')$ under $\mathcal{F}$ canonically.

III. X and X' have canonically isomorphic quantum cohomology rings under $\mathcal{F}$.

IV. Deformation/Decomposition: under generic symplectic perturbations, the deformed f can be decomposed into composite of ordinary $\mathbb{P}^r$-flops for various r's.

2.1 A Kähler Heuristic

For c_1-equivalent manifolds, we may select arbitrary Kähler metrics ω and ω' with volume 1 on X and X' respectively. Then

$$-\partial\bar{\partial}\log(\phi^*\omega)^n = -\partial\bar{\partial}\log(\phi'^*\omega')^n + \partial\bar{\partial}f,$$

which simplifies to $(\phi'^*\omega')^n = e^f(\phi^*\omega)^n$. I.e. the two degenerate metrics $\phi^*\omega$ and $\phi'^*\omega'$ have quasi-equivalent volume forms (same rate of degeneracy along the degenerate loci $E \subset Y$).

Question: Can one **rotate** $\phi^*\omega$ to $\phi'^*\omega'$ through (not-necessarily Kähler) degenerate metrics g_t while keeping the rate of volume degeneracy stable? Notice that

$$H^k(X) \cong L^k_2(X,\omega) = L^k_2(Y,\phi^*\omega).$$

A Kähler Candidate for this rotation:

Solve a family of complex Monge-Amperè equations via Yau's solution to the Calabi conjecture:

$$(\tilde{\omega} + \partial\bar{\partial}\varphi_t)^n = e^{t(f+c(t))}(\phi^*\omega)^n,$$

where $\tilde{\omega}$ is an arbitrary Kähler metric with volume 1 on Y and $c(t)$ is a normalizing constant at time t to make the right hand side has total integral 1 over Y. Let $\tilde{\omega}_t = \tilde{\omega} + \partial\bar{\partial}\varphi_t$.

Problem: Does $L_2^k(Y, \tilde{\omega}_0) = L_2^k(Y, \phi^*\omega)$?

Observation: c_1-equivalent implies "volume" equivalent.

2.2 A Motivic Approach

Let $\phi : Y = \mathrm{Bl}_Z X \to X$ with $Z \subset X$ smooth of codimension d, with exceptional divisor $E = \mathbb{P}_Z(N_{Z/X}) \to Z$, then

$$[E] = [Z](1 + \mathbb{L} + \cdots \mathbb{L}^{d-1}) = [Z][\mathbb{P}^{d-1}].$$

Localizing at $[\mathbb{P}^{d-1}]$, we get on the Grothendieck group $K_0(\mathrm{Var}_{\mathbb{C}})$,

$$[X] = [Y] - [E] + [Z] = ([Y] - [E]) + [E][\mathbb{P}^{d-1}]^{-1}.$$

For $\phi : Y \to X$ a composite of blowing-ups along smooth centers with $K_Y = \phi^* K_X + \sum_{i=1}^n e_i E_i$, $E := \bigcup_i E_i$ a NCD, we get a good CVF

$$[X] = \sum_{I \subset \{1,\ldots,n\}} [E_I^\circ] \prod_{i \in I} [\mathbb{P}^{e_i}]^{-1},$$

where $[E_I^\circ] := \bigcap_{i \in I} E_i \backslash \bigcup_{j \notin I} E_j$.

CVF for any birational morphism $\phi : Y \to X$ implies

$$X =_K X' \Longrightarrow [X] = [X'] \text{ in } S^{-1}K_0(\mathrm{Var}_{\mathbb{C}}),$$

where S is the multiplicative set generated by $\mathbb{P}^r$'s. Equivalently, $[P][X] = [P][X']$ for P a product of projective spaces.

Since $\chi_c(V) := \sum_{p,q} \chi_c^{p,q}(V)$ is not a zero divisor for smooth projective V, by applying the functor χ_c we conclude that X and X' have (non-canonically) isomorphic $\mathbb{Q}$-Hodge structures.

- p-adic integral for Betti numbers: Batyrev, Wang (1997).
- Motivic Integration: Kontsevich (1996), Denef-Loeser (1997).
- Weak Factorization Theorem: Wlodarczyk et. al., Y. Hu (1999).
- p-adic Hodge theory: Wang (2000), Ito (2001).

Chow motives:

Let $\mathcal{M}$ be category of motives. That is, smooth varieties with

$$\mathrm{Hom}_{\mathcal{M}}(\hat{X}_1, \hat{X}_2) = A^*(X_1 \times X_2).$$

For $U \in A^*(X_1 \times X_2)$, $V \in A^*(X_2 \times X_3)$, let $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$. The composition law is given by

$$V \circ U = p_{13*}(p_{12}^* U . p_{23}^* V).$$

U has induced maps on Chow groups and T-points $\mathrm{Hom}(\hat{T}, \hat{X}_i)$:

$$[U] : A^*(X_1) \to A^*(X_2); \quad a \mapsto p_{2*}(U . p_1^* a)$$

$$U_T : A^*(T \times X_1) \xrightarrow{U\circ} A^*(T \times X_2).$$

Identity Principle: $U = V$ if and only if $U_T = V_T$ for all T.

Proposition (Joint with H.-W. Lin) *For an ordinary* (r, r') *flip* $f : X \dashrightarrow X'$ *with* $r \le r'$, *the graph closure* $\mathcal{F} := \bar{\Gamma}_f$ *induces* $\hat{X} \cong (\hat{X}', p')$ *via* $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$, *where* $p' = \mathcal{F} \circ \mathcal{F}^*$ *is a projector.*

Proof. For any T, $\mathrm{id}_T \times f : T \times X \dashrightarrow T \times X'$ is also an ordinary (r, r') flip. To prove $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$, we only need to prove $\mathcal{F}^*\mathcal{F} = \mathrm{id}$ on $A^*(X)$ for any ordinary (r, r') flip. From

$$\mathcal{F}W = p'_*(\bar{\Gamma}_f . p^* W) = \phi'_* \phi^* W.$$

$$\phi^* W = \tilde{W} + j_*\big(c(\mathcal{E}).\bar{\phi}^* s(W \cap Z, W)\big)_{\dim W},$$

where $\mathcal{E}$ is defined by $0 \to N_{E/Y} \to \phi^* N_{Z/X} \to \mathcal{E} \to 0$.
key observation: the error term is lying over $W \cap Z$.

Let $W \in A_k(X)$. We may assume that $W \pitchfork Z$, so

$$\ell := \dim W \cap Z = k + (r + s) - (r + r' + s + 1) = k - r' - 1.$$

$\dim \phi^{-1}(W \cap Z) = \ell + r' = k - 1$, so $\phi^* W = \tilde{W}$ and $\mathcal{F} W = W'$.

Let B be an irred. component of $W \cap Z$ and $\bar{B} = \bar{\psi}(B) \subset S$ with dimension $\ell_B \le \ell$. $W' \cap Z'$ has components $\{B' := \bar{\psi}'^{-1}(\bar{B})\}_{B'}$.

Let $\phi'^* W' = \tilde{W} + \sum_{B'} E_{B'}$, where $E_{B'} \subset \bar{\phi}'^{-1} \bar{\psi}^{-1}(\bar{B})$, a $\mathbb{P}^r \times \mathbb{P}^{r'}$ bundle over $\bar{B}$. For the generic point $s \in \psi(\phi(E_{B'})) \subset \bar{B}$, we have

$$\dim E_{B',s} \ge k - \ell_B = r' + 1 + (\ell - \ell_B) > r'.$$

Since $r' \ge r$, $E_{B',s}$ contains positive dimensional fibers of ϕ. Hence $\phi_*(E_B) = 0$ and $\mathcal{F}^* \mathcal{F} W = W$. The proof is completed.

Corollary

Let $f : X \dashrightarrow X'$ be a (r, r') flip with $r \le r'$. If $\dim \alpha + \dim \beta = \dim X$*, then $\mathcal{F}$ is an orthogonal imbedding:*

$$(\mathcal{F}\alpha.\mathcal{F}\beta) = (\alpha.\beta).$$

Proof. We may assume that α, β are transversal to Z. Then

$$\begin{aligned}(\alpha.\beta) &= (\phi^*\alpha.\phi^*\beta) = ((\phi'^*\mathcal{F}\alpha - \xi).\phi^*\beta)\\ &= ((\phi'^*\mathcal{F}\alpha).\phi^*\beta) = (\mathcal{F}\alpha.(\phi'_*\phi^*\beta)) = (\mathcal{F}\alpha.\mathcal{F}\beta).\end{aligned}$$

Thus for ordinary flops, $\mathcal{F}^{-1} = \mathcal{F}^*$ in two senses.

Project: Do the similar argument on arc spaces $\mathcal{L}(X)$.

3. Chern Numbers, Complex Cobordism and the Weak Decomposition Theorem

3.1 Chern Numbers and Complex Elliptic Genera

Milnor and Novikov:
The complex cobordism class $[X] \in \Omega^U$ of a stable almost complex manifold X (i.e. $T_X \oplus \mathbb{R}_X$ is complex) is characterized by its Chern numbers. (Modulo cobordism by such manifolds with boundaries.)

An R-genus is a ring homomorphism $\varphi : \Omega^U \to R$. Equivalently, let $Q(x) \in R[\![x]\!]$. If $c(T_X) = \prod_{i=1}^n (1 + x_i)$, then

$$\varphi_Q(X) := \prod\nolimits_{i=1}^n Q(x_i)[X] =: \int_X K_Q(c(T_X))$$

defines an R-genus.

Write $Q(x) = x/f(x)$. The complex elliptic genera is defined by the three parameter ($k \in \mathbb{C}$, τ and a marked point z) power series

$$f(x) = \varphi_{k,\tau,z}(x) := e^{(k+\zeta(z))x} \frac{\sigma(x)\sigma(z)}{\sigma(x+z)}.$$

Totaro (Ann. 2000) showed that the most general Chern numbers invariant under $\mathbb{P}^1$-flops consists of the complex elliptic genera.

Hirzebruch reproved Totaro's theorem using Atiyah-Bott localization theorem. He showed that φ_Q is invariant under $\mathbb{P}^1$-flops if and only if $F(x) := 1/f(x)$ satisfies

$$F(x+y)(F(x)F(-x) - F(y)F(-y)) = F'(x)F(y) - F'(y)F(x)$$

and the solutions is given by the above $\varphi_{k,\tau,z}(x)$ exactly.

Theorem (—, J. Alg. Geom. 2003)

Complex elliptic genera are invariant under K-equivalence.

Corollary (Weak Decomposition Theorem)

In Ω^U, the ideal $\mathcal{I}_1 = \mathcal{I}_K$, where

$$\mathcal{I}_1 = \left\langle [X] - [X'] \,|\, X \text{ and } X' \text{ are related by a } \mathbb{P}^1 \text{ flop} \right\rangle,$$
$$\mathcal{I}_K = \left\langle [X] - [X'] \,|\, X =_K X' \right\rangle.$$

Remark. We expect symplectic deformations instead of complex cobordism. Also it is clearly not enough to consider only $\mathbb{P}^1$ flops for the general case.

3.2 A Change of Variable Formula for Chern Numbers

The most important step is to develop a change of variable formula for genera (or Chern numbers) under a single blowing-up $\phi : Y \to X$ along smooth center Z of codimension r.

Theorem (Residue Theorem)

For any $f(t) = t + \cdots$, $A(t) \in R[\![t]\!]$:

$$\int_Y A(E)\, K_Q(c(T_Y)) = \int_X A(0)\, K_Q(c(T_X)) + \int_Z \operatorname{Res}_{t=0}\left(\frac{A(t)}{f(t)\prod_{i=1}^r f(n_i - t)}\right) K_Q(c(T_Z)).$$

Here $c(N_{Z/X}) = \prod_{i=1}^r (1 + n_i)$ and the residue stands for the coefficient of the degree -1 term.

The residue term $= 0$ implies the CVF for one blowing-up. With f given, than for z not an r-torsion point we find the Jacobian factor

$$A(t, r) = e^{-(r-1)(k+\zeta(z))t} \frac{\sigma(t + rz)\sigma(z)}{\sigma(t + z)\sigma(rz)}.$$

The $r = 2$ case corresponds to functional equation

$$\frac{1}{f(x)f(y)} = \frac{A(x)}{f(x)f(y - x)} + \frac{A(y)}{f(y)f(x - y)}$$

which also has solutions given by f (and A is determined by f), but with z not a 2-torsion points. Thus

Complex elliptic genera are precisely the universal Chern numbers which admits a good CVF.

Theorem (Change of Variable formula)

Let $\varphi = \varphi_{k,\tau,z}$ be the complex elliptic genera and write $d\mu_X = K_\varphi(c(T_X))$. Then for any algebraic cycle D in X and birational morphism $\phi : Y \to X$ with $K_Y = \phi^* K_X + \sum e_i E_i$,

$$\int_D d\mu_X = \int_{\phi^* D} \prod_i A(E_i, e_i + 1)\, d\mu_Y.$$

Equivalently, $\phi_* \prod_i A(E_i, e_i + 1)\, d\mu_Y = d\mu_X$.

The proof is reduced to the blowing-up case by applying the weak factorization theorem.

The case $k = 0$ (elliptic genera) was also obtained by Borisov and Libgober using similar methods.

4. Invariance of Quantum Ring under Simple Ordinary Flops

(Joint with H-W. Lin, 2004)

4.1 Triple Product for Simple Flops

Let $f : X \dashrightarrow X'$ be a simple $\mathbb{P}^r$ flop. $h \subset Z$, $h' \subset Z'$ hyperplanes. $x = [h \times \mathbb{P}^r]$, $y = [\mathbb{P}^r \times h']$ in $E = \mathbb{P}^r \times \mathbb{P}^r$.

$$\phi^*[h^s] = x^s y^r - x^{s+1} y^{r-1} + \cdots + (-1)^{r-s} x^r y^s,$$
$$\mathcal{F}[h^s] = (-1)^{r-s}[h'^s].$$

For transversal $\alpha \in A^i(X)$: $\phi'^*\alpha' = \phi^*\alpha + (\alpha.h^{r-i})\,\dfrac{x^i - (-y)^i}{x+y}$.

Lemma

For simple $\mathbb{P}^r$-flops, $\alpha \in A^i(X)$, $\beta \in A^j(X)$, $\gamma \in A^k(X)$ with $i \le j \le k \le r$, $i + j + k = \dim X = 2r + 1$,

$$\mathcal{F}\alpha.\mathcal{F}\beta.\mathcal{F}\gamma = \alpha.\beta.\gamma + (-1)^r(\alpha.h^{r-i})(\beta.h^{r-j})(\gamma.h^{r-k}).$$

Quantum Corrections

The Poincaré pairing and the three point functions determine the quantum product.

$$\begin{aligned}\langle\alpha,\beta,\gamma\rangle &= \sum\nolimits_{\Gamma\in A_1(X)} \langle\alpha,\beta,\gamma\rangle_{0,3,\Gamma} \\ &= \alpha.\beta.\gamma + \sum\nolimits_{d\in\mathbb{N}} \langle\alpha,\beta,\gamma\rangle_{0,3,d\ell}\, q^{d\ell} \\ &\qquad + \sum\nolimits_{\Gamma\notin\mathbb{Z}\ell} \langle\alpha,\beta,\gamma\rangle_{0,3,\Gamma}\, q^{\Gamma}.\end{aligned}$$

For simple ordinary flops, let $U_d = R^1\pi_* e_4^* N_{Z/X}$, then

$$\begin{aligned}\langle\alpha,\beta,\gamma\rangle_{0,3,d} &= \int_{[\bar{M}_{0,3}(X,d\ell)]^{\mathrm{virt}}} e_1^*\alpha.e_2^*\beta.e_3^*\gamma \\ &= \int_{\bar{M}_{0,3}(\mathbb{P}^r,d)} e_1^*\alpha.e_2^*\beta.e_3^*\gamma.e(U_d).\end{aligned}$$

Deformations to the normal cone:

$\Phi : M \to \mathcal{X} = X \times \mathbb{P}^1$ be the blowing-up along $Z \times \{\infty\}$.
$M_t \cong X$ for $t \neq \infty$. $M_\infty = Y \cup \tilde{E}$ where
$\tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O})$,
$Y \to X = \mathcal{X}_\infty$ is the blowing-up along Z and
$Y \cap \tilde{E} = E = \mathbb{P}_Z(N_{Z/X})$.

Similarly $\Phi' : M' \to \mathcal{X}' = X' \times \mathbb{P}^1$ and $M'_\infty = Y' \cup \tilde{E}'$.
By construction, $Y = Y'$ and $E = E'$.

The key point is, when $S = \mathrm{pt}$, $\tilde{E} \cong \tilde{E}'$. A. Li and Y. Ruan's gluing formula or J. Li's degeneration formula implies the equivalence of $\langle \alpha, \beta, \gamma \rangle_{0,3,\Gamma}$ and $\langle \mathcal{F}\alpha, \mathcal{F}\beta, \mathcal{F}\gamma \rangle_{0,3,\mathcal{F}\Gamma}$ for $\Gamma \neq d\ell$.

If $\dim X = 3$, for simple $\mathbb{P}^1$-flops, the divisor axiom shows that

$$\begin{aligned}\sum\nolimits_d \langle \alpha, \beta, \gamma \rangle_{0,3,d\ell} &= \sum\nolimits_d (\alpha.d\ell)(\beta.d\ell)(\gamma.d\ell)\, \langle - \rangle_{0,0,d\ell}\, q^{d\ell} \\ &= (\alpha.\ell)(\beta.\ell)(\gamma.\ell) \frac{q^\ell}{1-q^\ell}\end{aligned}$$

via the **multiple cover formula** $\langle - \rangle_{0,0,d\ell} = 1/d^3$. Together with $(\mathcal{F}\alpha, \ell') = -(\mathcal{F}\alpha, \mathcal{F}\ell) = -(\alpha, \ell)$, then

$$\langle \mathcal{F}\alpha, \mathcal{F}\beta, \mathcal{F}\gamma \rangle - \langle \alpha, \beta, \gamma \rangle = -(\alpha.\ell)(\beta.\ell)(\gamma.\ell) \left(1 + \frac{q^{\ell'}}{1-q^{\ell'}} + \frac{q^\ell}{1-q^\ell} \right).$$

Under the identification $\ell' = -\ell$, this gives zero.

Theorem (Generalized Multiple Cover Formula)

For all $\alpha \in A^i(X)$, $\beta \in A^j(X)$, $\gamma \in A^k(X)$ with $i + j + k = 2r + 1$,

$$\langle \alpha, \beta, \gamma \rangle_{0,3,d\ell} = (-1)^{(d-1)(r+1)} (\alpha.h^{r-i})(\beta.h^{r-j})(\gamma.h^{r-k}).$$

Invariance of small quantum ring:

Since $(\mathcal{F}\alpha.h'^{(r-i)}) = (-1)^i(\mathcal{F}\alpha.\mathcal{F}h^{r-i}) = (-1)^i(\alpha.h^{r-i})$ etc.,

$$\langle \mathcal{F}\alpha, \mathcal{F}\beta, \mathcal{F}\gamma \rangle - \langle \alpha, \beta, \gamma \rangle = (-1)^r (\alpha.h^{r-i})(\beta.h^{r-j})(\gamma.h^{r-k})$$
$$- (\alpha.h^{r-i})(\beta.h^{r-j})(\gamma.h^{r-k}) \left(\frac{q^{\ell'}}{1 + (-1)^r q^{\ell'}} + \frac{q^{\ell}}{1 + (-1)^r q^{\ell}} \right).$$

Under $\ell' = -\mathcal{F}\ell$, this is -1 when r is odd and is 1 when r is even. Hence the right hand side cancels out, and it's done!

4.2 The Theory of Euler Data (After Lian-Liu-Yau)

We may represent the virtual fundamental class by the Euler class of the obstruction bundle

$$U_d = R^1\pi_* e_4^* N.$$

Here e_4 is the evaluation map and π is the forgetting map in

$$\begin{array}{ccc} \bar{M}_{0,4}(\mathbb{P}^r, d) & \xrightarrow{e_4} & \mathbb{P}^r \\ \downarrow{\scriptstyle \rho} & & \\ \bar{M}_{0,3}(\mathbb{P}^r, d) & & \end{array}$$

Then it is equivalent to proving that for all $d \in \mathbb{N}$, $i + j + k = 2r + 1$,

$$\int_{\bar{M}_{0,3}(\mathbb{P}^r, d)} e_1^* h^i . e_2^* h^j . e_3^* h^k . e(U_d) = (-1)^{(d-1)(r+1)}.$$

U_d is constructed on every $\bar{M}_{0,k}(\mathbb{P}^r, d)$ and is compatible under forgetting maps. $\operatorname{rk} U_d = (r+1)h^1(\mathbb{P}^1, \mathcal{O}(-d)) = (r+1)(d-1)$. $\dim M_{0,k}(\mathbb{P}^r, d) = (r+1)d + r + k - 3$. We may ask for

$$\int_{\bar{M}_{0,k}(\mathbb{P}^r,d)} e_1^* h^{i_1} \dots e_k^* h^{i_k}.e(U_d)$$

where $i_1 + \cdots + i_k = 2r + 1 + (k-3) = 2(r-1) + k$.

Let $\phi = \sum_{i=0}^r t_i h^i$, $e^k(\phi) = e_1^*\phi \dots e_k^*\phi$. Consider gluing sequence

$$b_d^k = e^k(\phi) e(U_d).$$

Let $N_d \cong \mathbb{P}^{(r+1)(d+1)-1}$ be the linear sigma model, $M_d^k = \bar{M}_{0,k}(\mathbb{P}^1 \times \mathbb{P}^r, (1,d))$ the non-linear sigma model and $\pi : M_d^k \to \bar{M}_{0,k}(\mathbb{P}^r, d)$ and $\zeta^k : M_d^k \to M_d^0 \equiv M_d \to N_d$.

$$Q_d^k = \zeta_*^k \pi^*(b_d^k).$$

Consider the $G = \mathbb{C}^\times \times (\mathbb{C}^\times)^{r+1} = \mathbb{C}^\times \times T$ action with weights $\alpha, \lambda_0, \ldots, \lambda_r$. Then the weights on N_d is $\lambda_i + s\alpha$ in coordinate

$$p_{is} = (0, \ldots, 0, z_{is} w_0^s w_1^{d-s}, 0, \ldots, 0).$$

Then $Q_d^k \in H_G^*(N_d) = \mathbb{Q}[\alpha, \lambda][\kappa]/f(\kappa)$, where $f(\kappa)$ is the Chern polynomial of $\mathbb{P}_G^r \to BG = (\mathbb{P}^\infty)^{r+1}$. For $i_{is} : p_{is} \to N_d$,

$$i_{p_{is}}^* \omega = \omega(\lambda_i + s\alpha).$$

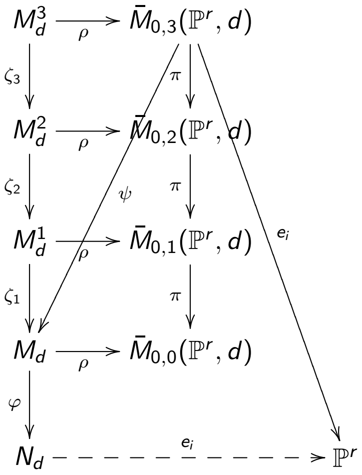
Consider $Q_d := \sum_{k=0}^\infty Q_d^k \dfrac{T^k}{k!} \in \mathcal{R}^{-1} H_G^*(N_d)[t_1, \ldots, t_r][\![T]\!]$.

Theorem

$\{Q_d\}$ is an $\Omega = e_T(N)^{-1}$-Euler data. That is,

$$i_{p_i}^*(\Omega)\, i_{p_{is}}^*(Q_d) = \overline{i_{p_{i0}}^*(Q_s)}\, i_{p_{i0}}^*(Q_{d-s})$$

for $s = 0, \ldots, d$, $i = 0, \ldots, r$.



ψ and φ are isomorphisms over smooth domain curves, $\bar{M}_{0,3}(\mathbb{P}^r, d)$ is the model making e_1, e_2 and e_3 becoming morphisms.

Recall that two Ω-Euler data $\{P_d\}$, $\{Q_d\}$ are linked if

$$i_{p_{j0}}^* P_d = i_{p_{j0}}^* Q_d$$

at $\alpha = (\lambda_j - \lambda_i)/d$ for all $i \neq j$ and $d > 0$.

To evaluate $i_{p_{j0}}^* Q'_d := \sum_{k=0}^3 i_{p_{j0}}^* Q_d^k \, T^k/k!$ at $\alpha = (\lambda_j - \lambda_i)/d$, we notice $Q_d^k(\lambda_j, (\lambda_j - \lambda_i)/d)$ is the restriction of $Q_d^k(\kappa, \alpha)$ to the smooth point $P_{ij} = (0, \ldots, w_0^d, \ldots, w_1^d, \ldots, 0) \in N_d$.

The uniqueness theorem in LLY says that two linked Ω-Euler data are indeed equal if for all i and d the following degree bound holds:

$$\deg_\alpha i_{p_{i0}}^*(P_d - Q_d) \leq (r+1)d - 2.$$

Theorem

$Q'_d := Q_d \pmod{T^4}$ is linked to, in fact equal to

$$P'_d = \sum_{k=0}^{3}\Big(\sum_{i=1}^{r} t_i\kappa^i\Big)^k \frac{T^k}{k!} \prod_{m=1}^{d-1}(-\kappa + m\alpha)^{r+1}$$
$$\equiv \exp\Big(\sum_{i=1}^{r} t_i\kappa^i T\Big) \prod_{m=1}^{d-1}(-\kappa + m\alpha)^{r+1} \pmod{T^4}.$$

In particular, in the non-equivariant limit we get

$$Q'_d|_{\alpha=0} = (-1)^{(r+1)(d+1)} e^{\sum t_i\kappa^i T} \kappa^{(r+1)(d-1)} \pmod{T^4}.$$

Pick out the coefficient of $t_i t_j t_k T^3$ by

$$\frac{\partial^3 Q'_d}{\partial t_i \partial t_j \partial t_k}\Big|_{\alpha=0,T=0} = (-1)^{(r+1)(d-1)} \kappa^{i+j+k} \kappa^{(r+1)(d-1)}$$

and notice that $\kappa^{(r+1)(d+1)-1} = 1$, we get the result.

4.3 Mukai Flops

Flopping contraction of Mukai type:
$\psi : (X, Z) \to (\bar{X}, S)$ with $N_{Z/X} = T^*_{Z/S} \otimes L$ for some $L \in \operatorname{Pic} S$.
Will construct the local model as a slice of ordinary flops with $F' = F^* \otimes L$.

$$\begin{array}{ccccc}
 & & \mathcal{E} = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F') \subset \mathcal{Y} & & \\
 & \swarrow^{\Phi} & \downarrow g & \searrow^{\Phi'} & \\
Z = \mathbb{P}_S(F) \subset \mathcal{X} & & & & Z' = \mathbb{P}_S(F') \subset \mathcal{X}' \\
 & \searrow_{\Psi} & & \swarrow_{\Psi'} & \\
 & & S \subset \bar{\mathcal{X}} & &
\end{array}$$

Suppose $\exists$ bi-linear map $F \times_S F' \to \eta_S$, $\eta_S \in \mathrm{Pic}(S)$.
$\mathcal{O}_{\mathbb{P}(F)}(-1) \to \bar{\psi}^* F$ pulls back to $\bar{\phi}^* \mathcal{O}_{\mathbb{P}(F)}(-1) \to \bar{g}^* F$, hence

$$\mathcal{Y} = \bar{\phi}^* \mathcal{O}_Z(-1) \otimes_{\mathcal{E}} \bar{\phi}'^* \mathcal{O}_{Z'}(-1) \to \bar{g}^*(F \otimes_S F') \to \bar{g}^* \eta_S.$$

$Y :=$ inverse image of the zero section of $\bar{g}^* \eta_S$ in $\mathcal{Y}$.
$X = \Phi(Y) \supset Z$, $X' = \Phi'(Y) \supset Z'$, $\bar{X} = g(Y) \supset S$ with restriction maps ϕ, ϕ', ψ, ψ'.

By tensoring the Euler sequence

$$0 \to \mathcal{O}_Z(-1) \to \bar{\psi}^* F \to \mathcal{Q} \to 0$$

with $\mathcal{S}^* = \mathcal{O}_Z(1)$ and notice that $\mathcal{S}^* \otimes \mathcal{Q} \cong T_{Z/S}$, we get by dualization

$$0 \to T^*_{Z/S} \to \mathcal{O}_Z(-1) \otimes \bar{\psi}^* F^* \to \mathcal{O}_Z \to 0.$$

The inclusion maps $Z \hookrightarrow X \hookrightarrow \mathcal{X}$ leads to

$$0 \to N_{Z/X} \to N_{Z/\mathcal{X}} \to N_{X/\mathcal{X}}|_Z \to 0.$$

$N_{X/\mathcal{X}}|_Z = \mathcal{O}(X)|_Z = \bar{\psi}^* \mathcal{O}(\bar{X})|_S$. Denote $\mathcal{O}(\bar{X})|_S$ by L. Recall $N_{Z/\mathcal{X}} \cong \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^* F'$. By tensoring with $\bar{\psi}^* L^*$, we get

$$0 \to N_{Z/X} \otimes \bar{\psi}^* L^* \to \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^*(F' \otimes L^*) \to \mathcal{O}_Z \to 0.$$

So $F' = F^* \otimes L$ if and only if $N_{Z/X} \cong T^*_{Z/S} \otimes \bar{\psi}^* L$.

For Mukai flops, namely $L \cong \mathcal{O}_S$, $F' = F^*$ with duality pairing $F \times_S F^* \to \mathcal{O}_S$. Consider $\pi : \mathcal{Y} \to \mathbb{C}$ via

$$\mathcal{Y} \to \bar{g}^* \mathcal{O}_S = \mathcal{O}_{\mathcal{E}} \cong \mathcal{E} \times \mathbb{C} \xrightarrow{\pi_2} \mathbb{C}.$$

We get a fibration with $\mathcal{Y}_t := \pi^{-1}(t)$, being smooth for $t \neq 0$ and $\mathcal{Y}_0 = Y \cup \mathcal{E}$. $E = Y \cap \mathcal{E}$ restricts to the degree $(1, 1)$ hypersurface over each fiber along $\mathcal{E} \to S$. Let $\mathcal{X}_t$, $\mathcal{X}'_t$ and $\bar{\mathcal{X}}_t$ be the proper transforms of $\mathcal{Y}_t$ in $\mathcal{X}$, $\mathcal{X}'$ and $\bar{\mathcal{X}}$.

For $t \neq 0$, all maps in the diagram

$$\begin{array}{ccccc} & & \mathcal{Y}_t & & \\ & \swarrow & & \searrow & \\ \mathcal{X}_t & & & & \mathcal{X}'_t \\ & \searrow & & \swarrow & \\ & & \bar{\mathcal{X}}_t & & \end{array}$$

are all isomorphisms. For $t = 0$ this is the Mukai flop. Thus Mukai flops are limits of isomorphisms. They preserve all interesting invariants like diffeomorphism type, Hodge type (Chow motive via $[Y] + [\mathcal{E}]$) and quantum rings etc. In fact all quantum corrections are zero. **END**