ANALYTIC CONTINUATIONS OF QUANTUM COHOMOLOGY UNDER ORDINARY FLOPS

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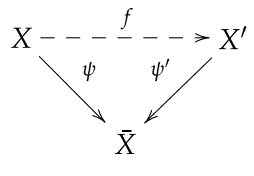
ABSTRACT. For ordinary flops over a smooth base, we determine the defect of cup product under the canonical correspondence and show that it is corrected by the small quantum product attached to the extremal ray. If the flop is of splitting type, the big quantum cohomology ring is also shown to be invariant after an analytic continuation in the Kähler moduli space.

This is a joint work with Yuan-Pin Lee (U. of Utah) and Hui-Wen Lin (Taiwan U.), which generalizes our previous similar result on simple flops [3].

0.1. **Background.** Two complex manifolds X and X' are *K-equivalent*, denoted by $X =_K X'$, if there are proper birational morphisms $(\phi, \phi') : Y \to X \times X'$ such that $\phi^* K_X = \phi'^* K_{X'}$. Major examples come from *birational minimal models* in Mori theory and especially from *birational Calabi-Yau manifolds* in the mathematical study of string theory. K-equivalent projective manifolds share the same Betti and Hodge numbers. It has been conjectured that a *canonical correspondence* $T \in A(X \times X')$ exists which induces isomorphisms of cohomology groups and preserves the *Poincaré pairing*. For a survey, see [7].

However, simple examples shows that the classical cup product is generally not preserved under $\mathscr{F}$. Since cohomology product corresponds to correlations of fields in quantum field theory, it is expected that *quantum product* would be *more natural* than the cup product among K-equivalent manifolds.

Flops are typical examples of K-equivalent birational maps:



In fact they form the building blocks to connect birational minimal models. The simplest flop is the simple P^1 flop (Atiyah flop) in dimension 3. It is known that up to deformations it generates, *locally* or *symplectically*, all K-equivalent maps for threefolds. The quantum corrections by extremal ray invariants to the cup product in the local 3D case was first noticed by Witten [8] and later globalized by Li-Ruan through the degeneration formula [6].

The higher dimensional generalizations are known as *ordinary P^r flops* (also abbreviated as "ordinary flops" or "P^r flops"). The local geometry is encoded by (S, F, F') where S is a smooth variety and F, F' are two rank $r+1$ vector bundles over S. If $Z \subset X$ is the f-exceptional loci, then $\bar\psi : Z \cong P(F) \to S \subset \bar X$ with fibers spanned by the flopped curve $C \cong P^1$ and $N_{Z/X} = \bar\psi^* F' \otimes \mathscr{O}_Z(-1)$. Similar structure holds for $Z' \subset X'$.

The study of invariance of quantum product under ordinary flops in higher dimensions was started in [3]. The canonical correspondence is given by the graph closure $T = [\bar\Gamma_f]$ and the quantum invariance under $\mathscr{F} = T_* : QH(X) \to QH(X')$ is proved for all *simple P^r flops*, i.e. with $S = \mathrm{pt}$. The crucial idea behind is to interpret *$\mathscr{F}$-invariance* in terms of *analytic continuations in Gromov-Witten theory*.

Let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of stable maps from genus g nodal curves with n marked points to X, and let $e_i : \overline{M}_{g,n}(X, \beta) \to X$ be the evaluation maps. The Gromov-Witten potential

$$F_g^X(t) = \sum_{n,\beta} \frac{q^\beta}{n!} \langle t^n \rangle_{g,n,\beta}^X = \sum_{n\geq 0,\, \beta\in NE(X)} \frac{q^\beta}{n!} \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n e_i^* t$$

is a formal function in $t \in H(X)$ and Novikov variables q^β, with $\beta \in NE(X)$, the Mori cone of effective classes of one cycles. *Modulo convergence issue*, it is a function on the *complexified Kähler cone* $\omega \in \mathcal{K}_X^{\mathbb{C}} := H_{\mathbb{R}}^{1,1} + i\mathcal{K}_X$ via

$$q^\beta = e^{2\pi i (\beta.\omega)}.$$

F_g^X and $F_g^{X'}$ share the same variable $t \in H \cong H(X, \mathbb{C}) \cong H(X', \mathbb{C})$ under $\mathscr{F}$, but different variables in $NE(X)$ and $NE(X')$. In the formal level $\mathscr{F} q^\beta = q^{\mathscr{F}\beta}$. But for $\ell = [C]$ (resp. $\ell' = [C']$) being the flopped curve classes,

$$\mathscr{F}\ell = -\ell'$$

which is not effective. By duality this implies that $\mathcal{K}_X^{\mathbb{C}} \cap \mathcal{K}_{X'}^{\mathbb{C}} = \emptyset$ in $H^2_{\mathbb{C}}$, hence F_g^X and $F_g^{X'}$ have different *domains* and any comparison of them could make sense only after analytic continuations over $\overline{\mathcal{K}_X^{\mathbb{C}} \cup \mathcal{K}_{X'}^{\mathbb{C}}} \subset H^2_{\mathbb{C}}$.

Let $\{T_i\}$ be a basis of H with $\{T^i\}$ being the dual basis with respect to the Poincaré pairing. Denote by $t = \sum t^i T_i$. The *big quantum ring* $(QH(X), *)$ uses only the genus zero potential with 3 or more marked points:

$$T_i *_t T_j = \sum_k \frac{\partial^3 F_0^X}{\partial t^i \partial t^j \partial t^k}(t) T^k = \sum_{n\geq 0,\, \beta\in NE(X)} \frac{q^\beta}{n!} \langle T_i, T_j, T_k, t^n \rangle_{0,n+3,\beta}^X T^k.$$

The *Witten-Dijgraff-Verlinde-Verlinde equation* (WDVV) guarantees that $*_t$ is a family of associative products on H parameterized by $t \in H$. This in tern equips H a structure of *formal linear Frobenius manifold H_X* with integrable (= flat) *Dubrovin connection*

$$\nabla^z = d - z^{-1} \sum_i dt^i \otimes T_i *_t$$

on the tangent bundle $TH = H \times H$ with parameter $z \in \mathbb{C}^\times$.

There is a natural embedding of $\mathcal{K}_X^{\mathbb{C}}$ in H. In suitable choice of coordinates we have $q^\ell = e^{2\pi i t_\ell}$ with the Kähler constraint $\operatorname{Im} t_\ell > 0$. Since now $\mathscr{F} q^\ell = q^{-\ell'}$, $\{q^\ell, q^{\ell'}\}$ serve as an atlas for P^1, the compactification of $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$. This gives the formal H an analytic P^1 direction. In [3], for simple flops the structural constants $\partial^3_{ijk} F_0^X(t)$ for big quantum product are shown to be analytic (in fact algebraic) in q^ℓ. Moreover, $\mathscr{F}$ identifies H_X and $H_{X'}$ through analytic continuations over this P^1. Bases on this, in [2] the Frobenius structure is further exploited to conclude analytic continuations from F_g^X to $F_g^{X'}$ for all simple flops and for all $g \geq 0$.

0.2. **Main results.** This current work studies Gromov-Witten theory, mostly in $g = 0$, under flops over a non-trivial base. It inherits the basic structure developed in [3] for the simple case, with various technical improvements to handle the complexity arising from the geometry of (S, F, F').

The major steps are: (1) Determination of the *defect* of cup product, (2) *Quantum corrections* attached to the flopping extremal ray $\mathbb{N}\ell$, (3) Reduction to local models $X_{loc} = P(N_{Z/X} \oplus \mathscr{O})$ and $X'_{loc} = P(N_{Z'/X'} \oplus \mathscr{O})$ by *degeneration analysis*, (4) Further reduction to *quasi-linearity* via reconstruction and WDVV equation, and (5) Proof of quasi-linearity for *split flops* via *Birkhoff factorizations* (BF) and the *generalized mirror transform* (GMT).

We now give a brief outline of ideas involved in these steps.

(1) Let $\{t_i\}$ be a basis of $A(S)$ with dual $\{\hat{t}_i\}$. Let $h = c_1(\mathscr{O}_Z(1))$ and $H_k = c_k(Q_F)$ where $Q_F \to Z = P(F)$ is the universal quotient bundle. Similarly we define h' and H'_k on the X' side. The H_k's are of fundamental importance since

$$\mathscr{F} H_k = (-1)^{r-k} H'_k$$

and the dual basis of $\{t_i h^j\}$ in $A(Z)$ is given by $\{\hat{t}_i H_{r-j}\}$.

Theorem 0.1 (Topological defect). *Let $a_1, a_2, a_3 \in A(X)$ with $\sum \deg a_i = \dim X$. Then*

$$\begin{aligned}
&(\mathscr{F} a_1 . \mathscr{F} a_2 . \mathscr{F} a_3)^{X'} - (a_1.a_2.a_3)^X \\
&\quad = (-1)^r \times \textstyle\sum_{i_*, j_*} (a_1.\hat{t}_{i_1} H_{r-j_1})^X (a_2.\hat{t}_{i_2} H_{r-j_2})^X (a_3.\hat{t}_{i_3} H_{r-j_3})^X \\
&\qquad\qquad \times (s_{j_1+j_2+j_3-(2r+1)}(F + F'^*) t_{i_1} t_{i_2} t_{i_3})^S,
\end{aligned}$$

where s_i is the i-th Segre class.

(2) The stable map moduli for the extremal ray has a bundle structures over S:

$$\begin{array}{ccccc}
\overline{M}_{0,n}(P^r, d\ell) & \longrightarrow & \overline{M}_{0,n}(Z, d\ell) & \xrightarrow{e_i} & Z \\
 & & \Psi_n \big\downarrow & \swarrow \bar{\psi} & \\
 & & S & &
\end{array}$$

In this case, the Gromov-Witten invariants on X are reduced to twisted invariants on Z by certain obstruction bundles. We define the fiber integral

$$\Big\langle \prod_{i=1}^{n} h^{j_i} \Big\rangle_d^{/S} := \Psi_{n*}\Big(\prod_{i=1}^{n} e_i^* h^{j_i} \Big) \in A^{\mu}(S)$$

as a *$\bar{\psi}$-relative invariant over S*, a cycle of codimension $\mu := \sum j_i - (2r + 1 + n - 3)$. The absolute invariant is obtained by the pairing on S:

$$\langle t_1 h^{j_1}, \cdots, t_n h^{j_n} \rangle_d^X = \Big(\langle h^{j_1}, \cdots, h^{j_n} \rangle_d^{/S} . \prod_{i=1}^{n} t_i \Big)^S .$$

If $\mu = 0$ then the invariant reduces to the simple case. This happens for $n = 2$ since then $j_1 = j_2 = r$. Thus we may calculate *extremal functions* based on the 2-point case by *reconstruction*. To state the result, let

$$f(q) := \frac{q}{1 - (-1)^{r+1} q}$$

which satisfies the functional equation $f(q) + f(q^{-1}) = (-1)^r$.

For 3-point functions, we show that $W_\mu := \sum_{d \in \mathbb{N}} \langle h^{j_1}, h^{j_2}, h^{j_3} \rangle^{/S} q^d$ with $1 \le j_i \le r$ is in $A^\mu(S)[f]$ (polynomial in f) and independent of the choices of j_i's.

Theorem 0.2 (Quantum corrections)**.** *The function W_μ is the action on f by a Chern classes valued polynomial in the operator $\delta = q d/dq$. It satisfies*

$$W_\mu - (-1)^{\mu+1} W'_\mu = (-1)^r s_\mu (F + F'^*).$$

This implies that the topological defect is corrected by the 3-point extremal functions. The analytic continuation for $n \ge 4$ points follows by reconstruction.

(3) To compare GW invariants of non-extremal classes, the application of *degeneration formula* and *deformation to the normal cone* is well suited for ordinary flops with base S. It reduces the problem to local models $p : \tilde{E} = P(N \oplus \mathscr{O}) \to Z$, $p' : \tilde{E}' = P(N' \oplus \mathscr{O}) \to Z$ with induced flop $f : \tilde{E} \dashrightarrow \tilde{E}'$. The reduction has two steps. The first reduces problems to *relative local* invariants $\langle A \mid \varepsilon, \mu \rangle^{(\tilde{E}, E)}$ where $E \subset \tilde{E}$ is the infinity divisor. The second is a further reduction back to *absolute local* invariants, with possibly *descendent insertions* coupled to E (f-special type).

The local model $\bar{p} := \bar{\psi} \circ p : \tilde{E} \to S$ and the flop f are all over S, with simple case as fibers. In particular, kernel of $\bar{p}_* : N_1(\tilde{E}) \to N_1(S)$ is spanned by the p-fiber line class γ and $\bar{\psi}$-fiber line class ℓ. $\mathscr{F}$ is compatible with $\bar{p}$. Namely

$$\begin{array}{ccc} N_1(\tilde{E}) & \xrightarrow{\ \mathscr{F}\ } & N_1(\tilde{E}') \\ & {\scriptstyle \bar{p}_* \oplus d_2} \searrow \quad \swarrow {\scriptstyle \bar{p}'_* \oplus d'_2} & \\ & N_1(S) \oplus \mathbb{Z} & \end{array}$$

is commutative. Thus functional equation of a generating series $\langle A \rangle$ is equivalent to those of its various subseries (fiber series) $\langle A \rangle_{\beta_S, d_2}$ labelled by $NE(S) \oplus \mathbb{Z}$.

Theorem 0.3 (Degeneration reduction). *To prove $\mathscr{F}\langle \alpha \rangle^X \cong \langle \mathscr{F}\alpha \rangle^{X'}$ for all α, it is enough to prove the local case $f : \tilde{E} \to \tilde{E}''$ for descendent invariants of f-special type:*

$$\mathscr{F}\langle A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho \rangle^{\tilde{E}}_{\beta_S, d_2} \cong \langle \mathscr{F}A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho \rangle^{\tilde{E}'}_{\beta_S, d_2}$$

for any $A \in H^*(\tilde{E})^{\oplus n}$, $k_j \in \mathbb{N} \cup \{0\}$, $\varepsilon_j \in H^*(E)$, $\beta_S \in NE(S)$ and $d_2 \geq 0$.

(4) The degeneration reduction works for the higher genus case as well. But for $g = 0$ more can be said. We assume now $X = X_{loc} = \tilde{E}$. Since $X \to S$ is a double projective bundle, $H(X)$ is generated by $H(S)$ and the relative hyperplane classes h for $Z \to S$ and ξ for $X \to Z$. Applications of the *reconstruction* [4], by moving all the divisors h, ξ as well as ψ classes into the last insertion, reduces the problem to

$$\langle t_1, \ldots, t_{n-1}, t_n \tau_k h^j \xi^i \rangle^X_{\beta_S, d_2}$$

with $t_* \in H(S)$, $d_2 \in \mathbb{Z}$, where $k \neq 0$ only if $i \neq 0$.

Since f is an isomorphism outside Z, the appearance of non-trivial ξ with all other insertions from S seems to suggests the *term-wise equality* of such series on X and X'. This is true for simple flops, but it tuns out to be too näive for general S. The best we can hope is still up to analytic continuation:

Conjecture 0.4 (Quasi-linearity).

$$\mathscr{F}\langle t_1, \ldots, t_{n-1}, \tau_k a\xi \rangle^X_{\beta_S, d_2} \cong \langle t_1, \ldots, t_{n-1}, \tau_k \mathscr{F} a\xi \rangle^{X'}_{\beta_S, d_2}.$$

Ii is crucial that we may reduce all fiber series into the case with the last insertion being $\tau_k a\xi^i$ with $i \neq 0$. For if $d_2 \neq 0$, by the divisor axiom we get

$$\langle t_1, \ldots, t_{n-1}, a \rangle^X_{\beta_S, d_2} = \langle t_1, \ldots, t_{n-1}, a, \xi \rangle^X_{\beta_S, d_2} / d_2.$$

If $d_2 = 0$, the reduction is achieved by a series of delicate applications of the WDVV equation.

(5) So far everything works for general bundles F and F'. In the last step we work out the *quasi-linearity* for ordinary flops of *splitting type*, namely $F \cong \bigoplus_{i=0}^r L_i$ and $F' \cong \bigoplus_{i=0}^r L_i'$ for some line bundles L_i and L_i' on S.

Theorem 0.5 (Main theorem). *The quasi-linearity holds for local ordinary flops of splitting type. Hence the big quantum cohomology ring is invariant under ordinary flops of splitting type up to analytic contiunations.*

The splitting assumption allows to apply the $\mathbb{C}^\times$ localizations technique along the fibers of the toric bundle $X \to S$. Recall that the big J function $J^X(z^{-1}; \tau) = 1 + \tau/z + O(z^{-2})$ with $\tau \in H(X)$, is the generating function of all genus zero GW invariants with at most one descendent insertion.

A recent result of Brown [1] says that certain localization data, the *hypergeometric modification*

$$I^X(z, z^{-1}; t^1, t^2, t) := \sum e^{\frac{t^1 h + t^2 \xi}{z}} I_\beta^{X/S}(z, z^{-1}) \bar{\psi}^* J_{\beta_S}^S(z^{-1}; t)$$

with $t \in H(S)$, lies in *Givental's Lagrangian cone* generated by $J^X(z^{-1}; \tau)$.

Based on the fact that the fiber cohomology are generated by divisor classes, we translate it (using big quantum differential equations) into

$$J^X(z^{-1}; \tau) = B(z; \partial) I^X(z, z^{-1}; t^1, t^2, t),$$

where $B(z; \partial) = 1 + O(z)$ is a differential operator in t^1, t^2 and t which removes the z-polynomial part of I^X in the $NE(X)$-adic topology. This process is a modification of the usually known Birkhoff factorization, and the change of variables

$$\tau = t + t^1 h + t^2 \xi + \cdots$$

is the generalized mirror transform (GMT) which equates the z^{-1} term.

Our proof of the quasi-linearity is based on the *explicit symmetry* of $I_\beta^{X/S}$ under $\mathscr{F}$ and through direct calculations via BF and GMT. Interesting *renormalization* phenomenon occurs in our proof. In this talk I will give explicit examples to demonstrate the renormalization procedure on GMT.

REFERENCES

[1] J. Brown; *Gromov-Witten invariants of toric fibrations*. arXiv:0901.1290.

[2] Y. Iwao, Y.-P. Lee, H.-W. Lin and C.-L. Wang; *Invariance of Gromov-Witten theory under simple flops*, preprint 2008. arXiv:0804.3816.

[3] Y.-P. Lee, H.-W. Lin and C.-L. Wang; *Flops, motives and invariance of quantum rings*, Ann. of Math., to appear.

[4] Y.-P. Lee and R. Pandharipande; *A reconstruction theorem in quantum cohomology and quantum K-theory*, Amer. J. Math. **126** (2004), 1367-1379.

[5] J. Li; *A degeneration formula for GW-invariants*, J. Diff. Geom. **60** (2002), 199–293.

[6] A.-M. Li and Y. Ruan; *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. **145** (2001), 151-218.

[7] C.-L. Wang; *K-equivalence in birational geometry*, in "Proceeding of the Second International Congress of Chinese Mathematicians (Grand Hotel, Taipei 2001)", math.AG/0204160.

[8] E. Witten; *Phases of $N = 2$ theories in two dimensions*, Nuclear Physics **B403** (1993), 159–222.

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