2006 12/20 at IMS, CUHK

Invariance of big quantum ring under flops

(joint with Y. P. Lee and H. W. Lin)

Motivations:

X, X' sm proj / $\mathbb{C}$, $X =_K X'$ "K-equivalent" if

$\exists$ $X \xleftarrow{\varphi} Y \xrightarrow{\varphi'} X'$, st. $\varphi^* K_X = \varphi'^* K_{X'}$

φ, φ' birat'l morphisms

eg. birat'l C-Y's or more gen'l birat'l min. models.

Known: $h^{p,q}(X) = h^{p,q}(X')$, but not ring str.

Q.1 (conj): $\exists$ canonical correspondence $\mathcal{F} \in A^n(X \times X')$ st. $\mathcal{F}: H(X) \xrightarrow{\sim} H(X')$?

"canonical" means $(\mathcal{F}\alpha, \mathcal{F}\beta)_{X'} = (\alpha, \beta)_X$

Q.2 (conj): Under $\mathcal{F}$, have isom quantum coh rings $QH(X) \xrightarrow{\sim} QH(X')$ in the sense of analytic continuations over the Kähler moduli.

explanations:

$\{T_i\}$ basis of $H(X)$, $T = \sum t_i T_i$, $\omega \in \mathcal{K}_X$ Kähler cone

$$\langle \alpha_1, \cdots, \alpha_n \rangle_{g,n,\beta} = \int_{[M_{g,n}(X,\beta)]^{virt}} e_1^*\alpha_1 \cdots e_n^*\alpha_n$$

$$\Phi^X(\omega, T) := \sum_{n=0}^{\infty} \sum_{\beta \in NE(X)} \frac{1}{n!} \langle T, \cdots, T \rangle_{0,n,\beta}\, e^{-2\pi(\omega,\beta)}, \quad e^{-2\pi(\omega,\beta)} =: q^\beta$$

big quantum product

$$T_i * T_j := \sum_k \Phi_{ijk} T^k$$

$\{T^k\}$ dual basis under $(\,,\,)_X$.

Since $\mathcal{K}_X \cap \mathcal{K}_{X'} = \phi$, $\{\mathcal{K}_{X'} \mid X' =_K X\}$ form a chamber str.

Each X gives a coor system $H(X)$ of a fixed H

$\mathcal{F}: H(X) \xrightarrow{\sim} H(X')$ as a (linear) transition fcn

Then Φ^X_{ijk} can be analytically conti from $\mathcal{K}_X$ to $\mathcal{K}_{X'}$ and agrees with $\Phi^{X'}_{ijk}$.

Date
Note

equivalently:

$\Phi_{ijk}(w,T)$ is well-defined on $\mathcal{K}_X \cup \mathcal{K}_{X'}$ with functional equation: $\mathcal{F}\Phi_{ijk}(w,T) = \Phi_{ijk}(w, \mathcal{F}T)$ (↑ change coor.)

History:

here $e^{-2\pi(w,\beta)_X} = e^{-2\pi(\mathcal{F}w, \mathcal{F}\beta)_{X'}}$ ($\mathcal{F}w \equiv w$)

(for β a flopped curve $\mathcal{F}\beta = -\beta'$)

$= e^{2\pi(w,\beta')_{X'}}$ $\Rightarrow \mathcal{F}(q^{\beta}) = q^{-\beta'}$

has convergence prob. if $w \in \mathcal{K}_{X'}$.

Def^{n}: simple $\mathbb{P}^r$-flop

$X \supset Z \simeq \mathbb{P}^r$ with $N_{Z/X} = \mathcal{O}(-1)^{\oplus r+1}$

then $E \subset Y = Bl_Z X$, $E \simeq \mathbb{P}^r \times \mathbb{P}^r$, $N_{E/Y} = \mathcal{O}(-1,-1)$

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow & \downarrow\varphi & \searrow & \\ Z \subset X & & \dashrightarrow_{f} & & X' \end{array}$$

φ' = contraction along another direction.

Lemma: $\alpha_i \in H^{2l_i}(X)$, $l_1 + l_2 + l_3 = \dim X = 2r+1$

h = hyp class of $Z = \mathbb{P}^r$, then

$(\mathcal{F}\alpha_1, \mathcal{F}\alpha_2, \mathcal{F}\alpha_3) = (\alpha_1,\alpha_2,\alpha_3) + (-1)^r (\alpha_1, h^{r-l_1})(\alpha_2, h^{r-l_2})(\alpha_3, h^{r-l_3})$

reverse

Lemma: $\mathcal{F} = [\Gamma_f] \in A^{2r+1}(X \times X')$.

For 3-folds, conj 2 solved by A. Li and Y. Ruan ~ 2000

3 ingredients:

(1) symplectic deformation and decomp of K-equiv map into composite of $\mathbb{P}^1$ flops.

(2) multiple cover formula for $\mathbb{P}^1 = C \subset X$, $N_{C/X} \cong \mathcal{O}(-1)^{\oplus 2}$

(3) relative G-W inv and degeneration formula.

for (1). Kawamata + Kollár + Friedman

(2). Witten ~ 1992: the classical product defect is corrected by 3-point functions on extr ray C.

(3) if $\beta \neq C$, then $\langle \alpha_1, \cdots, \alpha_n \rangle_{g,n,\beta} = \langle \mathcal{F}\alpha_1, \cdots, \mathcal{F}\alpha_n \rangle_{g,n,\mathcal{F}\beta}$

Date
Note $Z \cong \mathbb{P}^r$, ℓ = line class

We make progresses on (2) and (3):

Thm: (generalized multiple cover formula), for $\mathbb{P}^r$-flops. $\alpha_1, \cdots, \alpha_n \in H^{2\ell_i}(X)$, $\sum \ell_i = 2r+1 + (n-3)$, then

$$\langle \alpha_1, \cdots, \alpha_n \rangle_{0,n,d} \equiv \int_{\bar{M}_{0,n}(\mathbb{P}^r, d\ell)} e_1^* \alpha_1 \cdots e_n^* \alpha_n \cdot e(U_d)$$

obstruction bundle

$$= (-1)^{(d-1)(r+1)} N_{\ell_1 \cdots \ell_n} \underline{\underline{d^{n-3}}} (\alpha_1 . h^{r-\ell_1}) \cdots (\alpha_n . h^{r-\ell_n})$$

$N_{\ell_1 \cdots \ell_n}$ const. indep of d, and for $n=2,3$, $N_{\ell_1 \cdots \ell_n} \equiv 1$. recursively defined.

pf: based on LLY, Givental + Lee-Pandharipande (reconstruction thm)

Cor. the quantum product restricted to exceptional classes (extr. ray) are inv. under simple $\mathbb{P}^r$-flops.

Pf: since Poincaré pairing is preserved, we compare n-pt fcn's $n \geq 3$

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = (\alpha_1, \alpha_2, \alpha_3) + \sum_{d \in \mathbb{N}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{d\ell} q^{d\ell} + \sum_{\beta \notin \mathbb{Z}\ell} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_\beta q^\beta$$

modulo β with $\beta \notin \mathbb{Z}\ell$, get

$$(\mathcal{F}\alpha_i, h'^{(r-\ell_i)})_{X'} = (-1)^{\ell_i} (\mathcal{F}\alpha_i, \mathcal{F}h^{r-\ell_i}) = (-1)^{\ell_i} (\alpha_i, h^{r-\ell_i})_X$$

$$\langle \mathcal{F}\alpha_1 \mathcal{F}\alpha_2 \mathcal{F}\alpha_3 \rangle^{X'} - \langle \alpha_1, \alpha_2, \alpha_3 \rangle^X$$
$$= (\alpha_1, h^{r-\ell_1})(\alpha_2, h^{r-\ell_2})(\alpha_3, h^{r-\ell_3}) \left((-1)^r + \frac{(-1)^{2r+1} q^{\ell'}}{1-(-1)^{r+1} q^{\ell'}} - \frac{q^\ell}{1-(-1)^{r+1} q^\ell} \right)$$

under $\mathcal{F}$, $q^{\ell'} \longleftrightarrow q^{-\ell}$ ma the RHS $= 0$.

for $n = 3+k \geq 4$ $(k \geq 1)$:

$$\langle \alpha_1, \cdots, \alpha_n \rangle = N_{\ell_1 \cdots \ell_n} (\alpha_1 h_1^{r-\ell_1}) \cdots (\alpha_n h_n^{r-\ell_n}) \sum_{d=0}^{\infty} (-1)^{(d-1)(r+1)} d^k q^{d\ell}$$

$$= *** \left(q^\ell \frac{d}{dq^\ell} \right)^k \left(\frac{(-1)^{r+1}}{1-(-1)^{r+1} q^\ell} \right)$$

Since $q^{-\ell} \frac{d}{dq^{-\ell}} = - q^\ell \frac{d}{dq^\ell}$ and thee $(\cdots)_X$ and $(\cdots)_{X'}$ differ only by a constant, we get

$$\langle \mathcal{F}\alpha_1, \cdots, \mathcal{F}\alpha_n \rangle = \langle \alpha_1, \cdots, \alpha_n \rangle \quad \forall\, n \geq 4. \quad \square$$

under analytic continuation

Date

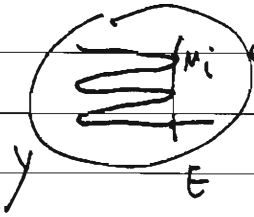
Note

Degeneration Analysis / reduction to local models.

rel. MV. (Y, E), $E \hookrightarrow Y$ sm div.

Top. type: $\Gamma = (g, n, \beta, \rho, \mu)$ $\quad \mu = (\mu_1, \cdots, \mu_\rho) \in \mathbb{N}^\rho$

$|\mu| = \sum \mu_i = (\beta . E)$



contact order $\quad A \in H(Y)^{\oplus n}$, $\varepsilon \in H(E)^{\oplus \rho}$

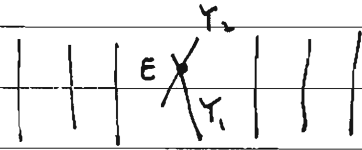
$$\langle A \mid \varepsilon, \mu \rangle^{(Y,E)} := \int_{[\overline{M}_\Gamma(Y,E)]^{virt}} e_Y^* A . e_E^* \varepsilon$$

evaluation: $e_Y : \overline{M}_\Gamma(Y,E) \to Y^n$, $e_E : \overline{M}_\Gamma(Y,E) \to E^\rho$

$f : X \overset{\psi \swarrow Y \searrow \psi'}{\dashrightarrow} X'$ simple $\mathbb{P}^r$-flop

Deformation to the normal cone to X and X':

$$W = Bl_{Z \times \{0\}} X \times \mathbb{A}^1$$
$$\Phi \downarrow$$
$$\mathcal{X} = X \times \mathbb{A}^1$$



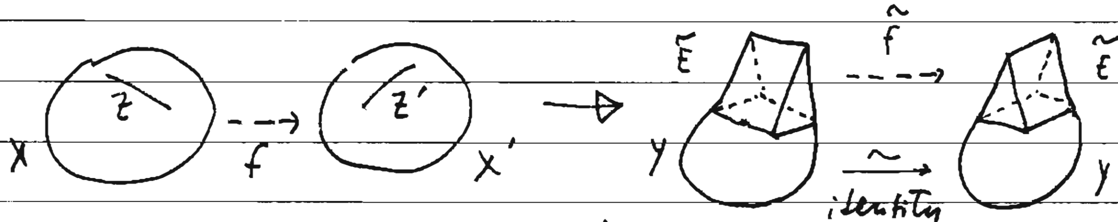
$W_t \cong X \quad W_0 = Y_1 \cup Y_2$

$t \neq 0$

Similarly $\Phi' : W' \to \mathcal{X}'$. $\quad Y \quad \tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O})$

$Y_1 = Y = Y_1'$, $Y_2 \cong \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(-1)^{r+1} \oplus \mathcal{O}) \cong Y_2'$

but the gluings along E are different:



Let $\{e_i\}$ basis of $H(E)$, $\{e^i\}$ dual basis

$\{e_I\}$ form a basis of $H(E^\rho)$, $|I| = \rho$, $e_I = e_{i_1} \otimes \cdots \otimes e_{i_\rho}$

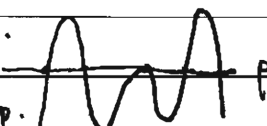
Then (Li-Ruan, J. Li, Ionel-Parker)

$$\langle \alpha \rangle^X_{g,n,\beta} = \sum_I \sum_{\eta = (\Gamma_1, \Gamma_2, I_\eta)} \frac{m(\mu)}{|\mathrm{Aut}\, \eta|} \langle \alpha(0) \mid e_I, \mu \rangle^{\bullet (Y,E)}_{\Gamma_1} \langle \alpha(0) \mid e^I, \mu \rangle^{\bullet (Y_2}_{\Gamma_2}$$

adm. triple st $\qquad$ not nec. conn. inv

$\Gamma_1 +_{I_\eta} \Gamma_2$ is conn. of type (g, n, β)

eg. 2 comp.



3 comp.

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Note

Lemma (coh. reduction)

Represent $\alpha(0) = (\alpha_1, \alpha_2)$, $({}^{\sigma}\mathcal{F}\alpha)(0) = (\alpha_1', \alpha_2')$

if $\alpha_1 = \alpha_1'$ then ${}^{\sigma}\mathcal{F}\alpha_2 = \alpha'$.

Prop A. To prove ${}^{\sigma}\mathcal{F}\langle\alpha\rangle^X \cong ({}^{\sigma}\mathcal{F}\alpha)^{X'}$, enough to show that

$${}^{\sigma}\mathcal{F}\langle A | \varepsilon, \mu\rangle \cong \langle {}^{\sigma}\mathcal{F} A | \varepsilon, \mu\rangle \quad \text{on } \tilde{E} \dashrightarrow \tilde{E}'.$$

where $\langle A | \varepsilon, \mu\rangle := \sum_{\beta_2 \in NE(E)} \frac{1}{|\mathrm{Aut}\,\mu|} \langle A | \varepsilon, \mu\rangle_{\beta_2}^{(\tilde{E}, E)} q^{\beta_2}$.

Rmk: if $\beta \in NE(X)$, $\beta(0) = (\beta_1, \beta_2)$ then $\beta_1 + \beta_2 = \varphi^* \beta$.

Prop B. For $\tilde{E} \dashrightarrow \tilde{E}'$, to prove ${}^{\sigma}\mathcal{F}\langle A | \varepsilon, \mu\rangle \cong \langle {}^{\sigma}\mathcal{F} A | \varepsilon, \mu\rangle$

$\forall A, (\varepsilon, \mu)$, enough to show

$${}^{\sigma}\mathcal{F}\langle A, \tau_{k_1}\varepsilon_1, \cdots, \tau_{k_p}\varepsilon_p\rangle \cong \langle {}^{\sigma}\mathcal{F}A, \tau_{k_1}\varepsilon_1, \cdots, \tau_{k_p}\varepsilon_p\rangle$$

i.e. for descendent inv. with ψ class only in E.

idea of pf: induction on $(|\mu|, n, p, (e_I, \mu))$

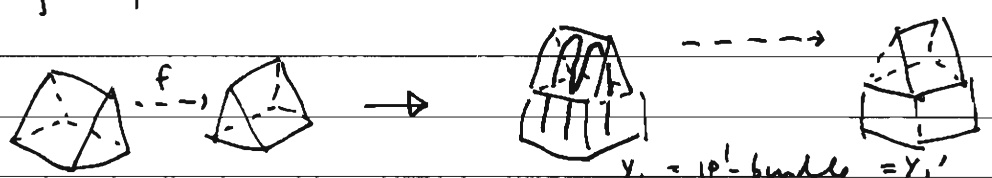
reverse ordering

Then $\langle \alpha_1, \cdots, \alpha_n, \tau_{\mu_1 - 1} e_{i_1} \cdots \tau_{\mu_p - 1} e_{i_p}\rangle^{\bullet \tilde{E}}$

$$= \sum_{\mu'} m(\mu') \sum_{I'} \langle \tau_{\mu_1 - 1} e_{i_1}, \cdots, \tau_{\mu_p - 1} e_{i_p} | e^{I'}, \mu'\rangle^{\bullet (Y_1, E)} \langle \alpha_1, \cdots, \alpha_n | e_{I'}, \mu'\rangle^{(\tilde{E}, E)} + R$$

fiber class integral — conn. inv — lower $|\mu|$

By applying deformation to normal cone to $Z \hookrightarrow \tilde{E}$.



$Y_1 = \mathbb{P}^1$-bundle $= Y_1'$

Key Point: $\exists!$ highest order term $= c(y) \langle \alpha_1, \cdots, \alpha_n | e_I, \mu\rangle^{(\tilde{E}, E)}$ ✱

Rmk: this is a more precise version of Maulik-Pandharipande.

For analytic continuation on local model:

Lemma (Quasi-linearity) ${}^{\sigma}\mathcal{F}\langle \tau_k \xi . \alpha\rangle^{\tilde{E}} = \langle \tau_k \xi' . {}^{\sigma}\mathcal{F}\alpha\rangle^{\tilde{E}'}$

both are finite sum. $\xi = [E] = \xi'$. (via LLY, Givental)

2 generator of Functional Equations: both uses → only here requires $q = 0$

GMCF + QL $\Rightarrow$ local case $\Rightarrow$ Thm. ✱