Change of Var for Gpx Genera and Its Applications to higher dim'l flops. 2001 Chin-Lung Wang

- Def^{n}: 2 Normal $\mathbb{Q}$-Gorenstein varieties (say, smooth) X, X' are K-equiv. $X =_K X'$ if $\exists$ diagram
$$\begin{array}{ccc} & Y & \\ {}^{\varphi}\swarrow & & \searrow^{\varphi'} \\ X & & X' \end{array}$$
with (smooth) Y st. $\varphi^* K_X = \varphi'^* K_{X'}$.

Problem: study relation of X and X' when $X =_K X'$.

Examples:
- X, X' birat'l Calabi-Yau manifolds
- X, X' birat'l minimal models.
  i.e. $K_X, K_{X'}$ $\mathbb{Q}$-Cartier, terminal sing. (canonical) and nef.

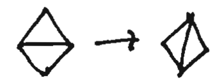
e.g. with
- X, X' bir'l $\mathbb{Q}$-Gorenstein. canonical sing.

$\mathrm{Ric}|_Z \leq 0$ $\leftarrow$ $X - Z \cong X' - Z'$, K_X nef on Z ($K_{X'}$ nef on Z')

$\mathrm{Ric}|_{Z'} \leq 0$.
- X, X' toric simplicial with same shed.

(Kawamata, Kollár): In dim 3, X, X' minimal bir'l $\Rightarrow$ $X \dashrightarrow X'$ can be decomposed into sequence of flops.

(Danilov): X, X' 3 dim'l toric mfd classified by Mori-Kollár.

shed(X) = shed(X') $\Rightarrow$ composite of "  "

Main Difficulty: DO NOT HAVE THIS IN DIM ≥ 4!

- Integration Formalism:

Basic idea: $K_Y = \varphi^* K_X + E = \varphi'^* K_{X'} + E'$

φ, φ' have same Jacobian factor E, E'!

$$\text{``}\quad \int_X d\mu_X = \int_Y J(E)\, d\mu_Y = \int_{X'} d\mu_{X'} \quad\text{''}$$

$\int_X d\mu_X$ = inv. of X, $\int_{X'} d\mu_{X'}$ = inv of (X').

Known Results: In any dim:
- $X =_K X'$, X, X' Proj. Sm. $\Rightarrow$ $h^{p,q}(X) = h^{p,q}(X')$

Batyrev for C-Y. [W-'97] in general. using p-adic int'l or motivic int'l.
- However, the ring str of X and X' are in general diff.

TODAY WE ARE INTERESTING IN CHERN NUMBERS, or cpx GENERA,

- R comm ring

  $Q(x) = 1 + \cdots \in R[[x]]$

  defines an R-genus φ

  Multiplicative sequence $\quad c = \prod(1+x_i)$

  $$K_Q(c) := \prod Q(x_i)$$

  then $\varphi(X) = K_Q(c(T_X))[X] = \int_X K_Q(c(T_X))$

  Topologically:

  $\Omega^U = \{\text{stably almost cpx mfd}\} / \text{cobordism by stably almost cpx mfd}$

  R-genus a ring homo

  $\varphi : \Omega^U \to R$ basically $\overset{\text{?}}{=}$ chern numbers

  Thm (Milnor): $X_1 \sim X_2$ in $\Omega^U_{\mathbb{Q}}$ $\Leftrightarrow$

  X_1, X_2 have the same chern numbers.

- complex elliptic genera: let $Q(x) = \dfrac{x}{f(x)}$

  $f(x) = x + \cdots$

  in algebraic coordinates:

  φ is parameterized by $(k, a, b, g_2) \in \mathbb{C}^4$ s.t.

  $$\frac{f'(x)}{f(x)} = -\frac{1}{2}\,\frac{\wp'(x) - b}{\wp(x) - a} + k$$

  where $\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3$ with g_3 defined by

  $$b^2 = 4a^3 - g_2 a - g_3$$

  in analytic coordinates: (k, w_1, w_2, z) $\left\{\begin{array}{l} k, \\ \text{affine Weierst eq'n} \\ \text{a marked pt} \end{array}\right.$

  $(a, b) = (\wp(z), \wp'(z))$, then

  $$f(x) = e^{(k + \zeta(z))x}\,\frac{\sigma(x)\,\sigma(z)}{\sigma(x+z)}$$

  $\left(\begin{array}{l} \zeta(x) = -\int^x \wp = \frac{1}{x} + \cdots \\ \sigma(x) = e^{\int^x \zeta} = x + \cdots \end{array}\right.$

  when $k = 0$, z a 2-torsion pt. get real elliptic genera

  $f(x) = 1/\sqrt{\wp(x) - \wp(z)}$.

• MAIN THEOREMS :

Thm 1. The most general χ genera that are preserved under the K-equiv. relation consist of precisely the χ elliptic genera.

Thm 2 : Let $I_F \trianglelefteq \Omega^U_{\mathbb{Q}}$ the ideal gen by all $[X]-[X']$ where X, X' are related by a classical flop. in 3-dim'l $\mathcal{O}(-1,-1)$ curve; in general $X-Z \cong X-Z'$

Let $I_K \trianglelefteq \Omega^U_{\mathbb{Q}}$ .... by X, X' with $X =_K X'$.

Then $I_F = I_K$. That is, up to χ cobordism, any K-equiv can be decomp. into classical flops.

$$\mathbb{P}^1\ Z \rightarrow W \leftarrow Z'\ \mathbb{P}^1$$

• STRATEGY : (A change of variable formula :)

↪ Residue Thm :

$$\begin{array}{ccc} E & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

$E = \mathbb{P}_Z(N)$, blow-up.

$\mathrm{codim}(Z, X) = r$

Z sm. $N = N_{Z/X}$.

$c(N) = \prod_{i=1}^{r}(1+n_i)$.

for any $A(t) \in R[[t]]$:

$$\int_Y A(E)\, K_Q(c(T_Y)) = \int_X A(0)\, K_Q(c(T_X)) + \int_Z \mathrm{Res}_{t=0}\left(\frac{A(t)}{f(t)\prod_{i=1}^{r} f(n_i - t)}\right) K_Q(c(T_Z))$$

Think $K_Q(c(T_X))$ as $d\mu_X$ (some elliptic measure).

• Application I : Take $A(t) = f(t)$ so $A(0) = 0$:

$$\int_Y E \cdot \left[\frac{f(E)}{E} \cdot K_Q(c(T_Y))\right] = \int_Z \underbrace{\mathrm{Res}_{t=0}\left(\frac{1}{\prod_{i=1}^{r} f(n_i - t)}\right)} K_Q(c(T_Z))$$

$$\frac{f(E)}{E} \cdot K_Q(c(T_Y)) = K_Q(c(T_E))$$

$$\int_Y E \cdot [\ldots] = \int_E K_Q(c(T_E)), \quad E = \mathbb{P}_Z(N)$$

In order for φ_Q to be strictly multiplicative for $\mathbb{P}^{r-1}$ bundle

get fun'l eq'n : $\sum_{i=1}^{r} \prod_{i \neq j} \frac{1}{f(x_i - x_j)} = c$. $\quad c = \varphi(\mathbb{P}^r)$

for real genera $f(x)$ odd.

Hirzebruch : $\forall r \in \mathbb{N}\ \exists$ genus (signature)

$\forall r \in 2\mathbb{N}\ \exists$ real elliptic genera.

• Application II: change of var formula:

$\forall\, r = 2, 3, \cdots$ $\exists\, A(t,r)$ in t as Jacobian factor st. $A(0,r) = 1$

$$\int_X K_\varphi(c(T_X)) = \int_Y \underline{A(E,r)} \cdot K_\varphi(c(T_Y))$$

$\Longleftrightarrow$ Functional Eq'n: $\exists\, A(t,r)$ ~~so~~ and $f(t)$ st.

$$\mathrm{Res}_{t=0}\left(\frac{A(t,r)}{f(t)\prod_{i=1}^{r} f(n_i - t)}\right) = 0 \quad \text{for all } n_i.$$

The case $r = 2$: ($E \to Z$ a $\mathbb{P}^1$ bundle) get

$$*: \quad \frac{1}{f(x)f(y)} = \frac{A(x)}{f(x)f(y-x)} + \frac{A(y)}{f(y)f(x-y)}.$$

In fact lengthy computation (not known before?!) via OPE jointly with S. K. Yu.

in maple solve this $\Rightarrow$ up to 20 degrees. uniquely solvable in f_2, f_3, f_4 and a_1. if not 2 torsion f_5

- A is completely determined by f
- $\boxed{f(x) = e^{(k+\zeta(z))x} \dfrac{\sigma(x)\sigma(z)}{\sigma(x+z)}}$ (with z not a 2-torsion pts)

Once know the expression of f, it is not hard to guess what $A(t,r)$ is:

$$A(t,r) = e^{-(r-1)(k+\zeta(z))t} \cdot \frac{\sigma(t+rz)\sigma(z)}{\sigma(t+z)\sigma(rz)}$$

Check: the residue term is an elliptic function in t: (for z not an r-torsion pt)

use $\prod_{j=1}^{r} \dfrac{\sigma(z-a_j)}{\sigma(z-b_j)}$ is elliptic $\Leftrightarrow$ $\Sigma a_i = \Sigma b_i$.

"$-rz + \sum_{i=1}^{r}(n_i + z) - z = \sum_{i=1}^{r} n_i - z$". ✻

Rmk: True replace X by D any cycle, Y by $\varphi^* D$.

<u>THEOREM</u>: CVF: for $\varphi: Y \to X$ st. $K_Y = \varphi^* K_X + \Sigma e_i E_i$, proper birational morphism, X, Y sm

then

$$\int_D \underbrace{K_\varphi(c(T_X))}_{d\mu_X} = \int_{\varphi^* D} \underbrace{\prod_i A(E_i, e_i+1)}_{\text{Jacobian}} \cdot K_\varphi\underbrace{(c(T_Y))}_{d\mu_Y}$$

Idea of pf:
- Induction $\exists$ if φ is a composite of blowing-ups.
- Apply weak factorization thm of Wlodarsczyk #
  also Abramovich-Karu-Matsuki and Wlo-.

Rmk: Lilgober & Borisov also prove this for $D = X$, $k = 0$.

Rmk: Totaro: Ann of Math (to appear)
most general chern numbers inv under classical flops
$\equiv$ cpx elliptic genera. (pf by rigidity thm of Hirzebruch, Höhn).

pf of thm 1: Now we prove that cpx elliptic genera actually are inv under K-equiv. by CVF. true for z not torsion, then use continuity to get all z.

pf of cpx elliptic genera Thm 2: $= \Omega^U_{\mathbb{Q}} / \text{classical flops} \twoheadleftarrow \Omega^U_{\mathbb{Q}} / \text{K-equiv}$ also surjective. So hence $\cong$

(Both Thm 1 & 2 follows from thm 3. *) $\langle \text{classical flops} \rangle = \langle \text{K-equivalence} \rangle$.

- ~~MAIN CONJECTURES~~: $X =_K X'$ sm. cpt. Kähler

(1) $T = \varphi'_* \circ \varphi^* : H^i(X) \to H^i(X')$ isom. of Hodge str.

(2) $\mathrm{Def}(X) \cong \mathrm{Def}(X')$

(3) (Y. B. Ruan) Under T, $QH(X) \cong QH(X')$.

?? (4) $\exists$ symplectic deformations of X and X' s.t. the K-equiv $X =_K X'$ decomposed into classical flops.

Rmk: True in dim 3, for (4), true up to cpx cobordism.

As we have seen, the most interesting thing is how the cpx elliptic genera come out from the Fcn. Eq'n (bec. then it's easy to guess the factor $A(t,r)$.)

CONJECTURE:

$$FE_r : \left\{ f, A \;\middle|\; \begin{array}{l} \dfrac{1}{\prod_{i=1}^{r} f(x_i)} = \displaystyle\sum_{j=1}^{r} \frac{A(x_j, r)}{f(x_j) \prod_{i \neq j} f(x_i - x_j)} \\ f(x) = x + \cdots \\ A(x) = 1 + \cdots \end{array} \right\} \equiv \left\{ \begin{array}{l} \text{Affine Weierstrass} \\ \text{Eq'n with a} \\ \text{marked point } z \\ \text{not an } r\text{-torsion} \end{array} \right\}$$

$r = 2$ case is done here.
$r = 3$ (Jeu-Kang Yu), In general only have $\supseteq$.

Rmk: Classical Thm of Weierstrass: fcn eq'n of $f(x), f(y), f(x+y) \Rightarrow$ elliptic fcn.
But FE_r not of this type.

End