Aspects on Calabi–Yau moduli

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Distance

For a polarized family of Calabi–Yau n-folds $\pi : \mathfrak{X} \to S$,

$$\omega_{WP} = c_1(F^n, \langle\, , \rangle) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\langle\Omega, \Omega\rangle$$

where Ω is any local holomorphic section of $F^n = \pi_* K_{\mathfrak{X}/S}$

Theorem (W-1996)

Let $\mathscr{H} \to \Delta^\times$ be a polarized VHS of weight n with $\operatorname{rank} F^n = 1$.
Then $0 \in \Delta$ is at finite distance if and only if

$$NF^n_\infty = 0.$$

Here $F^\bullet_\infty$ is the limiting Hodge filtration and N is the nilpotent part of the monodromy operator.

Assume that $S = (\Delta^\times)^r \times \Delta^m, 0 \in \bar{S} = \Delta^{r+m}$, and

$$D = \bar{S} \setminus S = D_1 \cup \cdots \cup D_r$$

a NCD with nilpotent monodromy N_j along $D_j = (t_j)$.

Conjecture

The point $0 \in \bar{S}$ is at finite g_{WP} distance if and only if

$$N_j F_\infty^n = 0, \qquad j = 1, \ldots, r.$$

Here $F_\infty^\bullet$ is the limiting Hodge filtration with respect to $N = \sum_{j=1}^r N_j$.

$\Leftarrow$ is easy: for $\gamma(t) = (t^{d_1}, \ldots, t^{d_r}, c_1, \ldots, c_m), d_j > 0$,

$$N_{\vec{d}} = \sum_{j=1}^r d_j N_j$$

all define the same weight filtration $W_\bullet$.

Approach I: VMHS

By the nilpotent orbit theorem, we may pick

$$\Omega(t) = e^{\frac{1}{2\pi\sqrt{-1}}\sum(\log t_j)N_j}a(t) \in F_t^n,$$

where $a(t)$ is holomorphic in t with $a(0) \in F_\infty^n$.

Theorem (T.-J. Lee 2016)

Let $\pi : \mathfrak{X} \to S = (\Delta^\times)^2 \times \Delta^m$ be a polarized family of Calabi–Yau 3-folds. Then the distance measured by the dominant term of the Weil–Petersson potential is infinite if $N_j F_\infty^3 \neq 0$ for some $j \in \{1, 2\}$.

Question

Is there a geodesic $\gamma \subset S$ towards $0 \in \bar{S}$ which lies in a holomorphic curve $C \subset S$? Is there a geodesic which fails this property?

Approach II: Hessian geometry

Given $ds^2 = \sum h_{ij}\, dy_i \otimes dy_j$, $h = -\log p$, i.e. $h_{ij} = p^{-2} p_i p_j - p^{-1} p_{ij}$.
Assume that $(p_{ij}) = D^2 p$ is invertible, $(dp)^2 := \sum p^{ij} p_i p_j$, then

$$h^{ij} = -p\left(p^{ij} - \frac{p^i p^j}{(dp)^2 - p}\right),$$

$$0 < \|\nabla h\|^2 = \sum h^{ij} h_i h_j = -\frac{1}{p}\left((dp)^2 - \frac{((dp)^2)^2}{(dp)^2 - p}\right) = \frac{(dp)^2}{(dp)^2 - p}.$$

Hence $(dp)^2 > p$, and $\|\nabla h\| \leq f \iff p \leq (1 - f^{-2})(dp)^2$. Then

$$|\gamma| = \int_\gamma \|\gamma'\|\, ds \geq \int_\gamma \frac{1}{f} |\nabla h \cdot \gamma'|\, ds \geq \left|\int_\gamma \frac{dh}{f}\right|.$$

Then $|\gamma| = \infty$ if $f = c$, or $f = c\,|h| \log |h| \log(\log |h|)$ etc..

Curvature

Let $\mathscr{H} \to S$ be an effective polarized VHS of weight n with $h^{n,0} = 1$. Let $g_{WP} = \sum g_{i\bar{j}}\, dt_i \otimes d\bar{t}_j$ on S.

Theorem (W-1997, see also Schumacher 1993)

The full curvature tensor of g_{WP} is given by

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}\, g_{k\bar{\ell}} + g_{i\bar{\ell}}\, g_{k\bar{j}}) + \frac{\langle \sigma_i \sigma_k\, \Omega, \sigma_j \sigma_\ell\, \Omega \rangle}{\langle \Omega, \Omega \rangle},$$

where $\sigma_i = \sigma(\partial/\partial t_i)$ is the infinitesimal period map.

For $n = 3$, it is equivalent to Strominger's formula

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}\, g_{k\bar{\ell}} + g_{i\bar{\ell}}\, g_{k\bar{j}}) + \textstyle\sum_{p,q} g^{p\bar{q}}\, F_{pik} \overline{F_{qj\ell}},$$

where F_{ijk} is the Bryant–Griffiths cubic form $F_{ijk} = \frac{\int_X \partial_i \partial_j \partial_k\, \Omega \wedge \Omega}{\int_X \Omega \wedge \bar{\Omega}}$.

Definition

The length of Yukawa coupling $\ell(\pi)$ for a VHS $\pi : \mathscr{H} \to S$ is the largest integer ℓ with $\sigma_{i_1} \cdots \sigma_{i_\ell} \not\equiv 0$ for some $i_1, \ldots i_\ell$.

(i) The existence of *maximal degenerate point* implies that $\ell(\pi) = n$. Then the VHS over S is rigid [Viehweg and Zuo].

(ii) There exist maximal families (moduli) of Calabi–Yau manifolds $\pi : \mathfrak{X} \to \mathscr{M}$ with $\ell(\pi) = 1$. Then

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}}).$$

That is, $(\mathcal{M}, g_{WP})$ is locally complex hyperbolic.

Question

Is there a mirror-like phenomenon near the boundary of $\mathscr{M}$?

Examples

Let $\mathfrak{M}_{n,m} \ni A = \{H_1, \ldots, H_m\}$ be the space of m-hyperplane arrangements of P^n in general positions. Let $n \geq 3$ be odd and

$$f_n : \mathfrak{X}_n \to \mathfrak{M}_{n,n+3}$$

be the family of $r = \frac{1}{2}(n+3)$-fold cyclic cover $\pi_A : X_A \to P^n$ branched along $H_A = \bigcup_{i=1}^{n+3} H_i \subset P^n$. Then

$$K_{X_A} = \pi_A^* K_{P^n} + (r-1)H_A \sim 0.$$

Theorem (Sheng–Xu–Zuo 2013)

There is a crepant resolution family $\tilde{f}_n : \tilde{\mathfrak{X}}_n \to \mathfrak{M}_{n,n+3}$, $\widetilde{X}_A \to X_A$, which is a maximal family of Calabi–Yau n-folds with $\ell(\tilde{f}_n) = 1$.

The S_n-Galois cover $\gamma : (P^1)^n \to \mathrm{Sym}^n P^1 = P^n$ induces

$$\Gamma : \mathfrak{M}_{1,n+3} \cong \mathfrak{M}_{n,n+3},$$

by $p_i \in P^1 \mapsto H_i = \gamma(\{p_I\} \times (P^1)^{n-1})$. Now $f_1 : \mathscr{C} = \mathfrak{X}_1 \to \mathfrak{M}_{1,n+3}$ is a family of curves C which are r-cyclic covers of P^1 at $n+3$ general points. Then $g(C) = \frac{1}{4}(n+1)^2$, $\dim \mathfrak{M}_{1,n+3} = n = h^{1,n-1}(\widetilde{X}_A)$, and

$$\begin{array}{ccc} \mathscr{C}^n/G & \xrightarrow{\sim} & \mathfrak{X}_n \ , \\ & {\scriptstyle h_n} \searrow \quad \swarrow {\scriptstyle f_n} & \\ & \mathfrak{M}_{n,n+3} & \end{array}$$

where $G = N \rtimes S_n$, N is abelian, and h_n is a $\mathbb{Z}/r$-Galois cover.

Question

Classify maximal Calabi–Yau families π with $\ell(\pi) = 1$. Is the harmonic K–S field $v \in (T^*)^{0,1} \otimes T \cong Hom(\bar{T}, T)$ corresponding to $\sigma_i \Omega$ parallel?

Singularities

Definition

A $\mathbb{Q}$-Gorenstein variety $\bar{X}$ has at most canonical singularities if there is a resolution $\phi : Y \to \bar{X}$ such that $K_Y = \phi^* K_X + \sum a_i E_i$ and $a_i \geq 0$.

Theorem (W-1996)

(i) *For a semi-stable CY degeneration $\mathfrak{X} \to \Delta$ with $\mathfrak{X}_0 = \bigcup_i X_i$, $NF^n_\infty = 0 \Longleftrightarrow$ there is a component X_0 in $\mathfrak{X}_0$ with $h^{n,0} \neq 0$.*

(ii) *Hence a CY degeneration $\pi : \mathfrak{X} \to \Delta$ with $\mathfrak{X}_0 = \bar{X}$ irreducible and with at most canonical singularities is at finite g_{WP} distance.*

Theorem (W-2003)

Let $\pi : \mathfrak{X} \to \Delta$ is a finite distance degeneration of CY manifolds. Assuming MMP, then up to a finite base change and birational modifications on the central fiber $\mathfrak{X}_0$ has only canonical singularities.

Idea of proof

- The s.s. reduction $\pi' : \mathfrak{X}' \to \Delta$ of π is also at finite distance. Write $\mathfrak{X}'_0 = \bigcup_{i=0}^N X'_i$ with X'_0 being the unique component with $0 \neq \Omega \in \Gamma(X'_0, K_{X'_0})$.
- Let $\pi'' : \mathfrak{X}'' \to \Delta$ be the relative minimal model of π' constructed from divisorial contractions and flips. The component X'_0 is not contracted, for otherwise it will be covered by extremal rational curves and $\kappa(X'_0) = -\infty$.
- The minimality of $\pi'' \implies K_{\mathfrak{X}''} \sim 0$. Also if $\mathfrak{X}''_0 = \sum_{i=0}^N X_i$ with $N > 0$ then $-K_{X_{i,\mathrm{red}}} = -\sum_{j \neq i} X_j|_{X_i}$ is anti-effective on $X_{i,\mathrm{red}}$ for each i. Hence the component X_0 corresponding to X'_0 does not have a non-zero section Ω.
- We conclude $\mathfrak{X}''_0 = X_0$ has at most canonical singularities.
- This special MMP was proved by Lai and Fujino in 2011.

Hausdorff convergence

Conjecture (W-1998, 2003)

(i) *CY degeneration with* $d_{WP} < \infty$

(ii) $\Longleftrightarrow_I$ *Continuity of* $\Omega(t)$ ($\Longleftrightarrow N_j F^n_\infty = 0 \Longleftrightarrow$ *canonical*)

(iii) $\Longleftrightarrow_{II}$ *CY family with uniformly bounded diameters.*

Theorem (Rong–Zhang 2011)

Let $(\mathfrak{X}, \mathscr{L}) \to \Delta$ *be a degeneration of polarized CY n-folds with* $K_{\mathfrak{X}/\Delta} \sim 0$. *Let* g_t, $t \in \Delta^\times$, *be the Yau metric in* $c_1(\mathscr{L}|_{\mathfrak{X}_t})$. *Then*

$$\operatorname{diam}_{g_t} \mathfrak{X}_t \leq 2 + c\,\langle \Omega(t), \Omega(t) \rangle.$$

This proves $\Longrightarrow_{II}$. Let $\mathcal{K}(n, c, V)$ be the class of projective manifolds with *volume* $= V$, $|\mathrm{Ric}| \leq 1$, and $\mathrm{Vol}\, B_r \geq c\, r^{2n}$ for any r-ball with $r \leq \operatorname{diam} X$. (Here $n = \dim_{\mathbb{C}} X$.)

[Tosatti 2015] *volume non-collapsing* $\iff$ *uniform boundedness of diameter* via Bishop volume comparison:

$$\frac{\mathrm{Vol}_{g_t} B_r}{c_n\, r^{2n}} \geq \frac{\int_{\mathfrak{X}_t} \omega_t^n}{c_n\, (\mathrm{diam}_{g_t} \mathfrak{X}_t)^{2n}}.$$

Theorem (Donaldson–Sun 2014)

(i) *Given n, V and c, $\exists\, k, N \in \mathbb{N}$ such that any $X \in \mathcal{K}(n, c, V)$ can be embedded in P^N by $\Gamma(X, L^{\otimes k})$.*

(ii) *Let $X_j \in \mathcal{K}(n, c, V)$ with Hausdorff limit X_∞. Then X_∞ is homeomorphic to a normal variety $W \subset P^N$. By passing to a subsequence and taking suitable projective transformations, we have $X_j \subset P^N$ converges to W in P^N which is klt.*

[Takayama 2015] then proved $\Longleftarrow_{II}$.

Metric completion

Conjecture

Given an irreducible component $\mathcal{M}$ of the moduli of polarized CY n-folds, there exist a finite number of $\mathcal{M}_{h_j}$'s such that

$$\mathcal{M}^c = \bigcup_j \mathcal{M}_{h_j} \supset \mathcal{M}$$

is complete w.r.t. the induced WP metric. Here $\mathcal{M}_{h_j}$ is an irreducible component of Viehweg's quasi-projective moduli of polarized CY with canonical singularities and with Hilbert polynomial h_j,

[Zhang 2014] Hausdorff completion using infinite covers.

Question (Main remaining question $\Longrightarrow_I$ for HD base)

Given a polarized CY degeneration $\mathfrak{X} \to S$ such that $p \in S$ is at finite WP distance. Do we have $\operatorname{diam} \mathfrak{X}_t < C$ *for t close to p?*

Transitions $Y \searrow X$ or $X \nearrow Y$ through $\bar{X}$

[Namikawa 2002] Let $S \to P^1$ be a rational elliptic surface with 6 singular fibers of type II (i.e., cuspidal). Then

- $\bar{X} = S \times_{P^1} S$ is a CY 3-fold with 6 cA_2 singular points:

$$x^2 - y^3 = u^2 - v^3.$$

- $\bar{X}$ admits smoothings to $X = S_1 \times_{P^1} S_2$ with $S_i \to P^1$ having disjoint discriminant loci, and
- an explicit small resolution $\pi : Y \to \bar{X}$ exists.
- The π-exceptional loci can not be deformed to a disjoint union of $(-1, -1)$-curves since a singular fiber of type II splits up into at most 2 singular fibers of type I, and a general fiber of small deformation of a singularity of $\bar{X}$, which preserves small resolutions, has 3 ODPs.

Conifold transitions

Theorem (S.-S. Wang 2015)

Namikawa's examples of extremal transition can be factorized into composition of conifold transitions up to flat deformations.

Lemma

Let $Y \to \bar{X}$ be a small resolution of a terminal 3-fold $\bar{X}$ and X a smoothing of $\bar{X}$. Then $e(Y) - e(X) \geq 2\,|\mathrm{Sing}(\bar{X})|$ with equality holds if and only if all the singularities of X are ODPs.

Proof: Let $C_i \to p_i \in \mathrm{Sing}(\bar{X})$. Since $\bar{X}$ is terminal Gorenstein, p_i is an isolated hypersurface singularity (cDV). It is well-known that

$$e(Y) - e(X) = \sum_i \mu_{p_i} + \sum (e(C_i) - 1)\,,$$

where μ_{p_i} = Milnor number. Supp C_i is a transverse union of n_i P^1's and $n_i = e(C_i) - 1$. So $\sum \mu_{p_i} + \sum n_i \geqslant 2\,|\mathrm{Sing}(X)|$, with "=" if and only if $n_i = \mu_{p_i} = 1$ for all i, i.e., p_i is an ODP. **QED**

Projective CICY web

Definition (Determinantal contractions/transitions)

Let $Y \subset S \times P^n$ be the zero loci of sections $s_i \in \Gamma(S \times P^n, \mathscr{L}_i)$, $i = 0, \ldots, n$, where $\mathscr{L}_i = L_i \boxtimes \mathscr{O}_{P^n}(1)$ with L_i semi-ample on S. We write $s_i = \sum_{j=0}^n s_{ij} x_j$, where $s_{ij} \in \Gamma(S, L_i)$. For $\pi : S \times P^n \to S$, let

$$\psi = \pi|_Y : Y \to \bar{X} := \pi(Y) \subset S.$$

For $p \in \bar{X}$, $\psi^{-1}(p)$ is not unique if and only if p is a singular point of

$$\Delta := \det s_{ij} = 0.$$

Since $\Delta \in \Gamma(S, \bigotimes_{i=0}^n L_i)$ and $\bar{X} = (\Delta)$, if $X_\tau = (\tau)$ is smooth for general $\tau \in \Gamma(S, \bigotimes_{i=0}^n L_i)$, then we get a transition $Y \searrow X_\tau$.

Theorem (Greene–Hubsch 1988, S.-S. Wang 2005)

The web of CICY 3-folds in $\prod P^{n_j}$ is connected by conifold transitions.

Invariance [LLW 2015]

Let $X \nearrow Y$ be a *projective* conifold transition of CY 3-folds through $\bar{X}$ with k ODPs $p_1, \ldots, p_k$, $\pi : \mathfrak{X} \to \Delta$, $\psi : Y \to \bar{X}$:

$$\begin{array}{ccccc} & & C_i \subset Y & & N_{C_i/Y} = \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \\ & & \downarrow \psi & & \\ N_{S_i/X} = T^*S^3 & S_i \subset X & \overset{\pi}{\rightsquigarrow} \; p_i \in \bar{X} & & \end{array}$$

Let $\mu := h^{2,1}(X) - h^{2,1}(Y) > 0$ and $\rho := h^{1,1}(Y) - h^{1,1}(X) > 0$.

$$\chi(X) - k\chi(S^3) = \chi(Y) - k\chi(S^2) \Longrightarrow \mu + \rho = k.$$

Hence there are non-trivial relations between the "vanishing cycles":

$$A = (a_{ij}) \in M_{k\times\mu}, \qquad \textstyle\sum_{i=1}^k a_{ij}[C_i] = 0,$$
$$B = (b_{ij}) \in M_{k\times\rho}, \qquad \textstyle\sum_{i=1}^k b_{ij}[S_i] = 0.$$

Let $0 \to V_{\mathbb{Z}} \hookrightarrow H_3(X,\mathbb{Z}) \to H_3(\bar{X},\mathbb{Z}) \to 0$ and $V := C_{\mathbb{Z}} \otimes \mathbb{C}$.

Theorem (Basic exact sequence)

We have an exact sequence of weight two pure Hodge structures:

$$0 \to H^2(Y)/H^2(X) \xrightarrow{B} \mathbb{C}^k \xrightarrow{A^t} V \to 0.$$

Since $\psi : Y \to \bar{X}$ deforms in families, this identifies $\mathscr{M}_Y$ as a codimenison μ boundary strata in $\mathscr{M}_{\bar{X}}$ and *locally* $\mathscr{M}_{\bar{X}} \cong \Delta^\mu \times \mathscr{M}_Y$. Write $V = \mathbb{C}\langle \Gamma_1, \ldots, \Gamma_\mu \rangle$ in terms of a basis Γ_j's. Then the α-periods

$$r_j = \int_{\Gamma_j} \Omega, \qquad 1 \leq j \leq \mu$$

form the *degeneration coordinates* around $[\bar{X}]$. The discriminant loci of $\mathscr{M}_{\bar{X}}$ is described by a central hyperplane arrangement $D_B = \bigcup_{i=1}^k D_i$:

Proposition (Friedman 1986)

Let $w_i = a_{i1}r_1 + \cdots + a_{i\mu}r_\mu$, then the divisor $D_i := \{w_i = 0\} \subset \mathcal{M}_{\bar{X}}$ is the loci where the sphere S_i shrinks to an ODP p_i.

- The β-periods in transversal directions are given by a function u:

$$u_p = \partial_p u = \int_{\beta_p} \Omega$$

- The BGY couplings extend over D_B and satisfy

$$u_{pmn} := \partial^3_{pmn} u = O(1) + \sum_{i=1}^{k} \frac{1}{2\pi\sqrt{-1}} \frac{a_{ip}a_{im}a_{in}}{w_i}$$

for $1 \le p, m, n \le \mu$. It is holomorphic outside this index range.

- Let $y = \sum_{i=1}^k y_i e_i \in \mathbb{C}^k$, with e^i's being the dual basis on $(\mathbb{C}^k)^\vee$. The trivial logarithmic connection on $\underline{\mathbb{C}}^k \oplus (\underline{\mathbb{C}}^k)^\vee \longrightarrow \mathbb{C}^k$ is

$$\nabla^k = d + \frac{1}{z} \sum_{i=1}^k \frac{dy_i}{y_i} \otimes (e^i \otimes e_i^*).$$

Theorem (Local invariance: $\mathrm{Exc}(\mathscr{A}) + \mathrm{Exc}(\mathscr{B}) = \mathrm{trivial}$)

(1) *∇^k restricts to the logarithmic part of ∇^{GM} on V^*.*

(2) *∇^k restricts to the logarithmic part of $\nabla^{\mathrm{Dubrovin}}$ on $H^2(Y)/H^2(X)$.*

Linked $A + B$ theory

Theorem (Lee–Lin–W 2015)

Let $[X]$ be a nearby point of $[\bar{X}]$ in $\mathscr{M}_{\bar{X}}$,

(1) *$\mathscr{A}(X)$ is a sub-theory of $\mathscr{A}(Y)$ (e.g. quantum sub-ring in genus 0).*

(2) *$\mathscr{B}(Y)$ is a sub-theory of $\mathscr{B}(X)$ (invariant sub-VHS).*

(3) *$\mathscr{A}(Y)$ can be reconstructed from a "refined $\mathscr{A}$ theory" on*

$$X^\circ := X \setminus \bigcup_{i=1}^k S_i$$

"linked" by the vanishing spheres in $\mathscr{B}(X)$.

(4) *$\mathscr{B}(X)$ can be reconstructed from the variations of MHS on $H^3(Y^\circ)$,*

$$Y^\circ := Y \setminus \bigcup_{i=1}^k C_i,$$

"linked" by the exceptional curves in $\mathscr{A}(Y)$.

For (3) and (4), effective methods are under developed.

Example (For (4), Lee–Lin 2016)

Conifold transitions $X \nearrow Y$ of CY 3-folds arising from toric degenerations:

$$\begin{array}{ccc} & & Y \subset \hat{P} = \hat{P}(2,4) \\ & & \big\downarrow \Psi \\ X \subset G = G(2,4) & \rightsquigarrow & P(2,4). \end{array}$$

- $\mathscr{B}(X) = \tau_G$ (tautological systems [Lian–Song–Yau 2013]), $\mathscr{B}(Y) = \tau_{\hat{P}}$ (extended GKZ [Lee–Lin 2016]).
- *For τ_G, the symmetry come from $\mathrm{SL}(4, \mathbb{C})$, which has $16 - 1 = 15$ dimensions. It consists of 12 roots and 3 torus action.*
- *For $\tau_{\hat{P}}$, its symmetry $\mathrm{Aut}^0(\hat{P})$ is generated by T^4 and 14 "roots"* [Cox 1995]*: for toric variety with fan Σ in $N_{\mathbb{R}}$, the roots $R(\Sigma, N)$ is given by $\{\alpha \in M \mid \exists p \in \Sigma_1, (\alpha, p) = -1, (\alpha, p') \geq 0 \quad \forall p' \neq p\}$.*
- *The 2 roots $\pm(1,1,1,1)$ are dropped since they move Ψ. The remaining 12 give those in τ_G. Thus $(\bigcup C_i, \tau_{\hat{P}}) \Longrightarrow \tau_G$.*

HAPPY 90th BIRTHDAY TO TSINGHUA MATH

Thank you for paying attention!